Jacobi Forms of Lattice Index

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Aim of the talk

1. Definition of Jacobi forms
2. Examples and parallels
3. Some results

I. What are Jacobi forms?

Some notation:

- As usual, $e_m(x) = e^{2\pi i x/m}$ and write $e(x)$ when $m = 1$.
- $\Gamma = \mathrm{SL}_2(\mathbb{Z})$, with elements $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.
- Upper-half plane: $\mathfrak{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$.
- The *weight* of a Jacobi form will be $k \in \mathbb{Z}_+$.

Apart from the weight, Jacobi forms also have an *index*. Some prerequisites:

- Denote by $\underline{L} = (L, \beta)$, where:
  - L is a *finite rank* $\mathbb{Z}$-module.
  - $\beta : L \times L \to \mathbb{Z}$ is symmetric, positive-definite, even $\mathbb{Z}$-bilinear form.

Remark

1. *Even* means $\beta(x,x) \in 2\mathbb{Z}, \forall x \in L$.
2. We denote $\beta(x) := \frac{1}{2}\beta(x,x)$.

- The *rank* of $\underline{L}$ is $\mathrm{rk}(\underline{L})$ (note: $L \simeq \mathbb{Z}^{\mathrm{rk}(\underline{L})}$).
- The *determinant* of $\underline{L}$ is $\det(\underline{L}) := \det(G)$, where
  - G is the *gram matrix* of L with respect to β: pick $\{e_i\}_{i=\overline{1,\mathrm{rk}(\underline{L})}}$ a $\mathbb{Z}$-basis for $L \implies G = (\beta(e_i, e_j))_{i,j}$.
  - Note this also gives $\beta(x,y) = \underline{x}^t G \underline{y}$.
- The *dual* of $\underline{L}$ is $L^{\#} := \{y \in L \otimes \mathbb{Q} : \beta(x,y) \in \mathbb{Z}, \forall x \in L\}$.
- The *level* of $\underline{L}$ is $\mathrm{lev}(\underline{L}) := \min_{\mathbb{N}_+}\{N : N \cdot \beta(y) \in \mathbb{Z}, \forall y \in L^{\#}\}$

What is the modular group?

More prerequisites: the *Heisenberg group* associated to $\underline{L}$ is $H_{\underline{L}}(\mathbb{Z}) := \{h = (x, y, 1) : x, y \in L\}$, with $hh' = (x + x', y + y', 1)$.

Remark

Γ acts on $H_{\underline{L}}(\mathbb{Z})$ from the right via $(x, y, 1)^A = ((x, y)A, 1)$.

Combine action of Γ and $H_{\underline{L}}(\mathbb{Z})$ to get

Definition (The Jacobi group associated to $\underline{L}$)

We define $J_{\underline{L}}(\mathbb{Z})$ to be the semi-direct product $\Gamma \ltimes H_{\underline{L}}(\mathbb{Z})$, with composition law:

$$(A, h)(A', h') = (AA', h^{A'} h').$$

More actions

- $J_{\underline{L}}(\mathbb{Z})$ acts on $\mathfrak{H} \times (L \otimes \mathbb{C})$ via $(A, h)(\tau, z) = \left(A\tau, \frac{z+x\tau+y}{c\tau+d}\right)$. We have a *modular* variable and an *elliptic* variable.
- $J_{\underline{L}}(\mathbb{Z})$ acts on $\mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}))$. If $\phi \in \mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}))$, then

$$\phi|_{k,\underline{L}}(A, h) := \left(\phi|_{k,\underline{L}}A\right)|_{k,\underline{L}}h,$$

where

$$\phi|_{k,\underline{L}}A(\tau, z) := \phi\left(A\tau, \frac{z}{c\tau+d}\right)(c\tau+d)^{-k}e\left(\frac{-c\beta(z)}{c\tau+d}\right)$$

and

$$\phi|_{k,\underline{L}}h(\tau, z) := \phi(\tau, z + x\tau + y)e(\tau\beta(x) + \beta(x, z)).$$

Jacobi forms

Definition (Jacobi forms of lattice index)

The space $J_{k,\underline{L}}$ of Jacobi forms of weight k and index $\underline{L}$ consists of all $\phi \in \mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}))$ that satisfy

1. $\phi|_{k,\underline{L}}(A,h) = \phi, \forall (A,h) \in J_{\underline{L}}(\mathbb{Z})$.
2. ϕ has a Fourier expansion of the form:

$$\sum_{\substack{n \in \mathbb{Z}, r \in L^{\#} \\ n \geq \beta(r)}} c(n,r) e(n\tau + \beta(r,z)).$$

- We have a 'modular interpretation': elliptic modular forms $f \in M_k(\Gamma)$ are in $1:1$ correspondence with functions $F(\Lambda_\tau)$ ($\Lambda_\tau = \mathbb{Z}\tau \oplus \mathbb{Z}$) satisfying $F(\lambda\Lambda_\tau) = \lambda^{-k}F(\Lambda_\tau)$, for all $\lambda \in \mathbb{C}^\times$ (Koblitz).
- Consider the following:
  - $\mathbb{H}_{\underline{L}}(\mathbb{Z})$ acts on $\mathfrak{H} \times (L \otimes \mathbb{C})$ via $h(\tau, z) = (\tau, z + x\tau + y)$.
  - This is properly discontinuous and fixed point free, so $\mathbb{H}_{\underline{L}}(\mathbb{Z}) \setminus (\mathfrak{H} \times (L \otimes \mathbb{C}))$ is an $\mathrm{rk}(\underline{L})-$ dimensional complex manifold $\mathcal{E}_{\underline{L}}$.

- The projection $\mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathfrak{H}$ induces a projection $\mathcal{E}_{\underline{L}} \to \mathfrak{H}$ whose fiber over τ is $(L \otimes \mathbb{C})/L\tau \oplus L \simeq (\mathbb{C}/\Lambda_\tau)^{\mathrm{rk}(\underline{L})} =: \mathcal{T}_{\tau,\underline{L}}$.
- Any $A \in \Gamma$ gives an isomorphism of tori $\left(\mathcal{T}_{\tau,\underline{L}}, 0\right) \simeq \left(\mathcal{T}_{A\tau,\underline{L}}, 0\right)$, induced by the map $z \mapsto \frac{z}{c\tau+d}$.
- Consider the action of Γ on $\mathfrak{H} \times (L \otimes \mathbb{C})$: $(\tau, z) \mapsto \left(A\tau, \frac{z}{c\tau+d}\right)$.
- Combine the actions of Γ and $\mathbb{H}_{\underline{L}}(\mathbb{Z})$ and set $\mathcal{A}_{\underline{L}} = J_{\underline{L}}(\mathbb{Z}) \setminus \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$. There is a projection $\mathcal{A}_{\underline{L}} \to \Gamma \setminus \mathfrak{H}$, whose fiber over τ is $\mathcal{T}_{\tau,\underline{L}}/\mathrm{Aut}\left(\mathcal{T}_{\tau,\underline{L}}, 0\right)$.

Conclusion

- Jacobi forms $\phi \in J_{k,\underline{L}}$ become functions $\Phi(L_\tau, z)$, with $L_\tau := L\tau \oplus L$ and $z \in \mathcal{T}_{\tau,\underline{L}} = (L \otimes \mathbb{C})/L_\tau$, which satisfy:

$$\Phi(L_\tau, z+\omega) = e(-\tau\beta(x) - \beta(x,z))\Phi(L_\tau, z),$$
$$\Phi(\lambda L_\tau, \lambda z) = \lambda^{-k} e(\lambda c \beta(z))\Phi(L_\tau, z),$$

  for $\lambda \in \mathbb{C}^\times$ and $\omega = x\tau + y \in L_\tau$.
- Elliptic modular forms can be interpreted as global sections of line bundles on the *modular curve* $\Gamma \setminus \mathfrak{H} \cup \{\text{cusps}\}$. Jacobi forms play a similar role for $J_{\underline{L}}(\mathbb{Z}) \setminus \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \cup \{\text{cusps}\}$.

This also gives:

$$J_{k,\underline{0}} \simeq M_k(\Gamma), \quad \text{for } \underline{0} = (L, 0).$$

Example (Jacobi theta functions associated to $\underline{L}$)

Fix $x \in L^{\#}$. Define:

$$\vartheta_{\underline{L},x}(\tau, z) := \sum_{\substack{r \in L^{\#} \\ r \equiv x \bmod L}} e\left(\tau\beta(r) + \beta(r, x)\right).$$

- These transform 'nicely' with weight $\mathrm{rk}(\underline{L})/2$ and index $\underline{L}$.
- They give isomorphism between spaces of Jacobi forms and spaces of *vector-valued Hilbert modular forms* (Boylan).
- Every Jacobi form has a *theta-expansion* (Ajouz). When $\mathrm{rk}(\underline{L})$ is odd, this gives a connection to *half-integral* elliptic modular forms.

Example

Fix $\underline{L} = (\mathbb{Z}, (x, y) \mapsto 2mxy)$, for $m \geq 0$. We get $J_{k,\underline{L}} = J_{k,m}$, studied extensively by Eichler and Zagier.

We get a connection to Siegel modular forms, because:

- Let Γ^J denote $J_{\underline{L}}(\mathbb{Z})$ for this particular choice of $\underline{L}$.

$$\Gamma \hookrightarrow \Gamma^J \hookrightarrow \mathrm{Sp}_2(\mathbb{Z}).$$

- Every SMF of degree 2 has a *Jacobi-Fourier expansion* (Piatetski–Shapiro). This also holds for degree $g > 2$, where now the expansion is in terms of Jacobi forms of *matrix index* $(g-1)$ (Bringmann).

From the work of Gritsenko:

- Modular forms *of orthogonal type* can be obtained as liftings of Jacobi forms. The former determine Lorentzian Kac–Moody Lie (super) algebra of Borcherds type.
- Jacobi forms are solutions to the *mirror symmetry* problem for $K3$ surfaces.
- For a compact complex manifold, one defines its elliptic genus, which can be a *weak* Jacobi form ($n \geq 0$).
- And much more...

Definition

Let $r \in L^{\#}/L$ and $D \in \mathbb{Q}_{\leq 0}$ be such that $\beta(r) \equiv D \bmod \mathbb{Z}$. We define

$$g_{\underline{L},r,D} := e(\tau(\beta(r) - D) + \beta(r, z)).$$

When $D < 0$, we define

$$P_{k,\underline{L},r,D} := \sum_{\gamma \in J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})} g_{\underline{L},r,D}|_{k,\underline{L}}\gamma$$

and, when $\beta(r) \in \mathbb{Z}$, let

$$E_{k,\underline{L},r} := \frac{1}{2} \sum_{\gamma \in J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})} g_{\underline{L},r,0}|_{k,\underline{L}}\gamma.$$

Why the interest?

- Both are elements of $J_{k,\underline{L}}$.
- Eisenstein series:
  - Perpendicular to *Jacobi cusp forms* $(n > \beta(r))$ with respect to a suitably defined *Petersson scalar product*.
  - We get a decomposition:

$$J_{k,\underline{L}} = S_{k,\underline{L}} \oplus J_{k,\underline{L}}^{Eis}.$$

  - Their *twists* by Dirichlet characters modulo N_x (level of x) form a *basis of eigenforms* of $J_{k,\underline{L}}^{Eis}$ with respect to (again) suitably defined *Hecke operators*.

- Poincaré series (previously undefined in this setting):
    - They are cusp forms.
    - They reproduce Fourier coefficients of other cusp forms via the Petersson scalar product.
    - Furthermore, they *generate* $S_k(\Gamma)$.
    - Our main interest is in *reproducing kernels* of linear operators defined between spaces of Jacobi cusp forms and elliptic modular forms.

Proposition

For any $\phi \in S_{k,\underline{L}}$,

$$\langle \phi, P_{k,\underline{L},r,D} \rangle = \lambda_{k,\underline{L},D} c(n,r),$$

where

$$\lambda_{k,\underline{L},D} := 2^{-2k+\frac{\mathrm{rk}(\underline{L})}{2}+2} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2} - 1\right) \det(\underline{L})^{-\frac{1}{2}} (\pi|D|)^{-k+\frac{\mathrm{rk}(\underline{L})}{2}+1}$$

and $c(n,r)$ is the Fourier coefficient of ϕ corresponding to $e(\tau(\beta(r) - D) + \beta(r,z))$.

Theorem

For $k > rk(\underline{L}) + 2$, $P_{k,\underline{L},r,D}$ is a cusp form. It has the following Fourier expansion:

$$P_{k,\underline{L},r,D}(\tau, z) = \sum_{\substack{n' \in \mathbb{Z}, r' \in L^{\#} \\ n' > \beta(r')}} G_{k,\underline{L},D,r}(n', r') e\left(n'\tau + \beta(r', z)\right),$$

where

$$\begin{aligned} G_{k,\underline{L},D,r}(n', r') :=& \delta_{\underline{L}}(D, r, D', r') + (-1)^k \delta_{\underline{L}}(D, r, D', -r') + 2\pi i^k \\ & \times \det(\underline{L})^{-\frac{1}{2}} \left(\frac{D'}{D}\right)^{\frac{k}{2} - \frac{rk(\underline{L})}{4} - \frac{1}{2}} \cdot \sum_{c \geq 1} \left(H_{\underline{L},c}(n, r, n', r')\right. \\ & \left. + (-1)^k H_{\underline{L},c}(n, r, n', -r')\right) \cdot J_{k - \frac{rk(\underline{L})}{2} - 1}\left(\frac{4\pi (DD')^{\frac{1}{2}}}{c}\right), \end{aligned}$$

where $D' = \beta(r') - n'$, we use $J_{k-\frac{rk(\underline{L})}{2}-1}(\cdot)$ for the Bessel function and

$$H_{\underline{L},c}(n, r, n', r') := c^{-\frac{rk(\underline{L})}{2} - 1} \sum_{\lambda(c)} e_c(\beta(r', \lambda + r)) K(n', \beta(\lambda) + \beta(r + \lambda) + n; c).$$

In the last equation, λ runs through a complete set of representatives of L/cL and $K(n', \beta(\lambda) + \beta(r + \lambda + n); c)$ is a Kloosterman sum.

2) Operators on the spaces of Jacobi forms

- Operators give *structure* to the space.
- They facilitate *equivariant lifts* between different types of modular forms.
- They have algebraic interpretations in terms of the surfaces that our modular forms underlie.
- In [Ajouz, 2015], we are given
  - Hecke operators:

$$T_0(l)\phi := l^{k-2-rk(\underline{L})} \sum_{\gamma \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} l^{-1} & 0 \\ 0 & l \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})} \phi|_{k,\underline{L}}\gamma,$$

  - Action of the *orthogonal group* of $\underline{L}$:

$$W(\alpha)\phi(\tau, z) = \sum_{n \in \mathbb{Z}, r \in L^{\#}} c(n, \alpha(r))e(n\tau + \beta(r, z)).$$

We want a theory of *newforms*. For that, we need:

Definition

We define the operator $U(l)$ on the space $J_{k,\underline{L}}$ by:

$$U(l)\phi(\tau, z) := \phi(\tau, lz).$$

Remark

1. The operator $U(l)$ corresponds to the endomorphism "multiplication by l" on $\mathcal{T}_{\tau,\underline{L}} = (L \otimes \mathbb{C}) / (L\tau \oplus L)$.
2. Think of $U(l) : M_k(N) \to M_k(lN)$,

$$U(l)f(\tau) = \sum a(ln)q^n.$$

Theorem

The operator $U(l)$ maps $J_{k,\underline{L}}$ to $J_{k,\underline{L}'}$, where $\underline{L}' = (L, \beta')$, where $\beta' = l^2\beta$. Moreover, if $\phi \in J_{k,\underline{L}}$ has the Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^{\#} \\ n\geq\beta(r)}} c(n,r)e(n\tau + \beta(r,z)),$$

then $U(l)\phi$ has the following Fourier expansion:

$$U(l)\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r'\in L^{\#'} \\ n\geq\beta'(r')}} c(n, lr')e(n\tau + \beta'(r', z)),$$

with the convention $c(n, lr') = 0$ unless r' is an $l-$th multiple of another element of $L^{\#'}$.

- Note that the level of $\underline{L}'$ is $\mathrm{lev}(\underline{L}') = l^2 \cdot \mathrm{lev}(\underline{L})$.

Definition

We define the operator $V(l)$ on the space $J_{k,\underline{L}}$ by:

$$V(l)\phi(\tau,z) = l^{\frac{k}{2}-1} \sum_{\substack{M \in \Gamma\backslash\mathcal{M}_2(\mathbb{Z}) \\ \det(M)=l}} U(\sqrt{l})\left(\phi|_{k,\underline{L}}M\right)(\tau,z).$$

Remark

1. Assume that L_τ is contained in L' with index l. If $\{\omega_1,\omega_2\}$ is a basis for L', then there exists $M \in \mathcal{M}_2(\mathbb{Z})$ with determinant l, such that $\left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right) = M\left(\begin{smallmatrix}\omega_1\\\omega_2\end{smallmatrix}\right)$.
2. If $M = \left(\begin{smallmatrix}a & b\\c & d\end{smallmatrix}\right)$, then $U(\sqrt{l})\left(\phi|_{k,\underline{L}}M\right)(\tau,z)$ contains a factor of $\phi\left(M\tau, \frac{lz}{c\tau+d}\right)$.

Theorem

The operator $V(l)$ maps $J_{k,\underline{L}}$ to $J_{k,\underline{L}''}$, where $\underline{L}'' = (L, \beta'')$, where $\beta'' = l\beta$. Moreover, if $\phi \in J_{k,\underline{L}}$ has the Fourier expansion

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^{\#} \\ n\geq\beta(r)}} c(n,r)e(n\tau + \beta(r,z)),$$

then $V(l)\phi$ has the following Fourier expansion:

$$V(l)\phi(\tau, z) = \sum_{\substack{n,r'' \\ n\geq\beta''(r'')}} \sum_{\substack{a|(n,l) \\ \frac{r''}{a}\in L^{\#''}}} a^{k-1} c\left(\frac{nl}{a^2}, \frac{lr''}{a}\right) e(n\tau + \beta''(r'', z)).$$

- Find isomorphisms of the type

$$J_{k,m} \simeq \mathfrak{M}^{-}_{2k-2}(m),$$

like in [Skoruppa & Zagier, 1988].

- Find decomosition of the type

$$S_{k,m} \bigoplus_{\substack{l,l' \\ l^2 l' | m}} V(l')U(l)S^{New}_{k,m/l^2l'},$$

like in [Eichler & Zagier, 1985].

Thank you!