ATKIN–LEHNER THEORY OF $\Gamma_1(N)$-MODULAR FORMS

ANDREEA MOCANU

1. Introduction

A.O.L Atkin $(1925 - 2008)$ was a British Mathematician who got his PhD at Cambridge under the supervision of J. Littlewood. He worked in Computational Number Theory and especially Coding and Cryptography before moving to the USA. J. Lehner $(1912 - 2013)$ was an American mathematician working on automorphic forms. Atkin–Lehner theory means the theory of *newforms* and *oldforms*.

Our goal is to find a canonical basis for $S_k(\Gamma_1(N))$. This is because a fixed space of modular forms $M_k(G)$ can be decomposed in the first instance into $M_k(G) = S_k(G) \bigoplus M_k(G)^{Eis}$. The first question one can ask is what are the two spaces in the decomposition orthogonal with respect to? Next, assuming that we know what $M_k(G)^{Eis}$ looks like (i.e. if we have a basis), we naturally focus our attention on S_k.

For $G = \Gamma_1(N)$ in particular, we can also decompose $M_k(G) =$ *oldforms* $\bigoplus$ *newforms*, where oldforms "come from" level $M \mid N$, while newforms appear for the first time in level N. A combination of these two techniques will help us reach our goal.

Our main tool will be the following variation of the *Spectral Theorem* from Linear Algebra, which we cite without proof:

Theorem 1.1 (Spectral Theorem)**.** *Let R be a commutative algebra of normal operators acting on a finite dimensional vector space V. Then there exists an orthogonal basis $\{f_1, \dots, f_n\}$ of V formed of eigenvectors for R.*

We would like to define normal operators from the start:

Definition 1.1 (Normal operator)**.** A linear operator T defined on a complex vector space is called *normal* if it commutes with its *adjoint* T^*, where T^* is defined by:

$$\langle Tf, g\rangle = \langle f, T^*g\rangle.$$

We begin to tackle each of the other notions we encounter in Theorem 1.1.

2. Modular forms

We first want to pin down the vector space V. For N a positive integer, remember the following congruence subgroups (i.e. they contain $\Gamma(N)$ for some N):

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1 \right\},$$
$$\Gamma_0(N) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} (\mathrm{mod} N) \right\} \cap \Gamma,$$
$$\Gamma_1(N) = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} (\mathrm{mod} N) \right\} \cap \Gamma.$$

Definition 2.1 (Modular form)**.** Let G be a congruence subgroup and $k \in \mathbb{N}$. Let v be a multiplier system of weight k on G (a homomorphism $G \to S^1$, such that $v(-I) = (-1)^k$ if $-I \in G$). We say that $f \in M_k(G, v)$ if

(i) f is holomorphic on $\mathfrak{H} \cup \{\text{cusps}\} = \overline{\mathfrak{H}}$.
(ii) $f|_k\gamma(\tau) = v(\gamma)f(\tau)$ for all $\gamma \in G$ and $\tau \in \overline{\mathfrak{H}}$. We define the $|_k$ action as

$$f|_k\gamma(\tau) = (c\tau + d)^{-k} f(\gamma\tau),$$

with G acting on the upper half-plane by linear fractional transformations.

We will denote the Fourier expansion of a modular form f as

$$f(\tau) = \sum_{n=0}^{\infty} a_n(f) q^n.$$

We denote $M_k(G) := M_k(G, 1)$ and $S_k(G, v)$ to be the space of *cusp forms*, i.e. those modular forms in $M_k(G, v)$ which vanish at the cusps of G. We would also like to define a generalization of the $|_k$ action, for any $\beta \in \mathrm{GL}_2^+(\mathbb{Q})$:

$$f|_k\beta(\tau) = \det(\beta)^{k-1}(c\tau + d)^{-k} f(\beta\tau).$$

Thus, our vector space V will be $S_k(\Gamma_1(N))$ for a fixed level N initially and, once we define newforms, it will be $S_k(\Gamma_1(N))^{new}$. It is well known that $M_k(\Gamma_1(N))$ is finite dimensional and dimension formulas can be found, for example, in [1].

3. The Petersson scalar product

Next, we need a scalar product. H. Petersson (1902 – 1984) was a German mathematician, who is most famous for the Ramanujan–Petersson conjecture about the Fourier coefficients of modular forms. He also introduced a *metric* on the upper half-plane.

The upper half-plane is a model for the hyperbolic plane, so it comes endowed with hyperbolic geometry. In particular, we can define a measure on $\mathfrak{H}$:

Definition 3.1 (Hyperbolic measure)**.** The hyperbolic measure on $\mathfrak{H}$ is:

$$d\mu(\tau) = \frac{dxdy}{y^2},$$

where $\tau = x + iy$.

The measure μ is invariant under automorphisms of $\mathfrak{H}$ (=$\mathrm{GL}_2^+(\mathbb{R})$), so in particular it is invariant under Γ: $d\mu(\gamma\tau) = d\mu(\tau)$ for all $\gamma \in \Gamma$. We can also extend it to $\overline{\mathfrak{H}}$, since the set of cusps is countable and hence measurable.

Definition 3.2 (Fundamental Domain)**.** A fundamental domain for the action of Γ on $\overline{\mathfrak{H}}$ is

$$D^* = \left\{\tau \in \mathfrak{H} : |\Re(\tau)| \leq \frac{1}{2}, |\tau| \geq 1\right\} \cup \{i\infty\}.$$

Furthermore, if G is a congruence subgroup, then a fundamental domain for the action of G on $\overline{\mathfrak{H}}$ is

$$X(G) = \bigcup_j \alpha_j D^*,$$

where $\Gamma = \bigcup_j \{\pm I\} G\alpha_j$, i.e. $\{\alpha_j\}_j$ is a set of coset representatives of $\{\pm I\}G \setminus \Gamma$.

Definition 3.3 (Volume of a congruence subgroup)**.** If G is a congruence subgroup, we define

$$V_G = \int_{X(G)} d\mu(\tau).$$

In particular, $V_\Gamma = \frac{\pi}{3}$ and $V_G = \frac{\pi}{3}[\Gamma : \{\pm I\}G]$.

Finally, we can define

Definition 3.4 (Petersson scalar product)**.** Let G be a congruence subgroup and $f, g \in M_k(G, \chi)$ for some χ, at least one of them a cusp form. We define

$$\langle f, g\rangle_G = \frac{1}{V_G}\int_{X(G)} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau).$$

We give a brief explanation for the integrating factor $f(\tau)\overline{g(\tau)}\Im(\tau)^k$: it is, first of all, invariant under the action of G on $\overline{\mathfrak{H}}$; secondly, it is bounded on $\overline{\mathfrak{H}}$, as a consequence of the fact that it is bounded on $X(G)$, which in turn is a consequence of boundedness on each $\alpha_j D^*$, provided at least one of f, g is a cusp form (to show the last statement, use the Fourier expansions of f and g and let $y \to \infty$). It turns out that $\langle f, g\rangle_G$ is in fact independent of G, due to the following:

Lemma 3.1. *If $G \subseteq G'$ are two congruence subgroups (which implies $M_k(G') \subseteq M_k(G)$), then*

$$\langle f, g\rangle_G = \langle f, g\rangle_{G'}.$$

Proof. (sketch) Since $G \subseteq G'$, we can take the set of coset representatives $\{\beta_i\}$ of $\{\pm I\}G \setminus \{\pm I\}G'$. It follows that a set of coset representatives for $\{\pm I\}G\setminus\Gamma = \{\pm I\}G\setminus\{\pm I\}G'\cdot\{\pm I\}G'\setminus\Gamma$ is $\{\gamma_{ij}\}_{i,j}$, where $\gamma_{ij} = \beta_i\alpha_j$, where the α_j's are coset representatives of $\{\pm I\}G' \setminus \Gamma$. Thus,

$$\begin{aligned}
\langle f, g\rangle_G &= \frac{1}{V_G}\int_{X(G)} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau) \\
&= \frac{1}{V_G}\int_{\cup_i\cup_j(\beta_i\alpha_j)D^*} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau) \\
&= \frac{1}{V_G}\sum_i \int_{\cup_j\beta_i(\alpha_j D^*)} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau) \\
&= \frac{1}{V_G}[G' : G]\int_{\cup_j\alpha_j D^*} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau) \\
&= \frac{1}{V_{G'}}\int_{X(G')} f(\tau)\overline{g(\tau)}\Im(\tau)^k d\mu(\tau) = \langle f, g\rangle_{G'}.
\end{aligned}$$

□

In particular, we can drop the $_G$ from the notation for the Petersson scalar product from now on.

4. Hecke operators

Our family of linear operators will be comprised of:

Definition 4.1 (Diamond operator)**.** For any $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, define the diamond operator

$$\langle d\rangle : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$$

acting on $M_k(\Gamma_1(N))$ as

$$\langle d\rangle : f \mapsto \langle d\rangle f := f|_k\alpha,$$

for any $\alpha = \left(\begin{smallmatrix} a & b \\ c & \delta \end{smallmatrix}\right) \in \Gamma_0(N)$ with $\delta \equiv d \bmod N$.

Definition 4.2 (T_p)**.** If p is a prime, define the p-th Hecke operator

$$T_p : M_k(\Gamma_1(N)) \to M_k(\Gamma_1(N))$$

acting on $M_k(\Gamma_1(N))$ as

$$T_p : f \mapsto T_p f := \begin{cases} \sum_{i=0}^{p-1} f|_k \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix}, & \text{if } p \mid N \\ \sum_{i=0}^{p-1} f|_k \begin{pmatrix} 1 & i \\ 0 & p \end{pmatrix} + f|_k \begin{pmatrix} m & n \\ N & p \end{pmatrix} |_k \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \end{cases}$$

where in the latter case $mp - nN = 1$.

We set ourselves the following goals:

1. To explain these definitions and why they work.
2. To show that these operators form a commutative family.
3. To extend the definitions of $\langle d\rangle$ and T_p to all $n \in \mathbb{N}$.
4. To show that these operators are normal.

If we succeed, we automatically get an orthogonal basis of eigenvectors for $S_k(\Gamma_1(N))$, in view of the previous sections.

4.1. **Double coset operators.** The operators $\langle d\rangle$ and T_p are in fact particular examples of *double coset operators*: let G_1 and G_2 be congruence subgroups and let $\alpha \in \mathrm{GL}_2^+(\mathbb{Q})$. We define the double coset

$$G_1\alpha G_2 = \{g_1\alpha g_2 : g \in G_1, g_2 \in G_2\}.$$

There is an action of G_1 on $G_1\alpha G_2$ given by left multiplication, so we can choose a set of coset representatives for the quotient space $G_1 \setminus G_1\alpha G_2 = \{\beta_j\}_j$, which is finite. This allows us to define an action on modular forms for G_1:

Definition 4.3 (Double coset operator)**.** Let G_1 and G_2 be congruence subgroups and let $\alpha \in \mathrm{G}_2^+(\mathbb{Q})$. Define the double coset operator action on $f \in M_k(G_1)$ as

$$f|_k G_1\alpha G_2(\tau) := \sum_j f|_k\beta_j(\tau),$$

where $G_1\alpha G_2 = \bigcup_j G_1\beta_j$.

This definition is independent of the choice of coset representatives β_j. The beauty of the double coset operator $G_1\alpha G_2$ is that:

Proposition 4.1. *The double coset operator $G_1\alpha G_2$ maps $M_k(G_1)$ to $M_k(G_2)$ and, furthermore, cusp forms to cusp forms.*

Proof. (sketch) We need to show invariance of $f|_k G_1\alpha G_2(\tau)$ under the action of G_2, holomorphicity and preservation of cusp forms. For $g_2 \in G_2$, we have

$$f|_k G_1\alpha G_2|_k g_2(\tau) = \sum_j f|_k\beta_j|_k g_2(\tau) = \sum_j f|_k(\beta_j g_2)(\tau)$$

and we claim that multiplication by g_2 simply permutes the coset representatives, i.e.

$$f|_k G_1\alpha G_2|_k g_2(\tau) = \sum_j f|_k\alpha_j(\tau) = f|_k G_1\alpha G_2(\tau),$$

for a different choice of coset representatives α_j of $G_1 \setminus G_1\alpha G_2$.

Since f is holomorphic, we claim that $f|_k\gamma$ is holomorphic for any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$ and so a finite linear combination of such terms is also holomorphic. Last but not least, if f is a cusp form for G_1, then $f|_k\gamma$ also vanishes at the cusps for any $\gamma \in \mathrm{GL}_2^+(\mathbb{Q})$. □

Going back to Definition 4.1, let us take $G_1 = G_2 = \Gamma_1(N)$ and $\alpha \in \Gamma_0(N)$. We remind the reader that $\Gamma_1(N) \subseteq \Gamma_0(N)$ and, furthermore, $\Gamma_1(N)$ is the kernel of the surjective homomorphism

$$\varphi : \Gamma_0(N) \to (\mathbb{Z}/N\mathbb{Z})^\times$$

given by $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \mapsto d \bmod N$. In other words, $\Gamma_1(N) \lhd \Gamma_0(N)$ and so $\Gamma_1(N) \setminus \Gamma_1(N)\alpha\Gamma_1(N) \simeq \alpha$. We thus get an action of $\Gamma_0(N)$ on $M_k(\Gamma_1(N))$, and since its subgroup $\Gamma_1(N)$ acts trivially, we can define an action of the quotient space $\Gamma_1(N) \setminus \Gamma_0(N) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$, which is precisely the diamond operator we defined.

To obtain T_p, take again $G_1 = G_2 = \Gamma_1(N)$, but this time take $\alpha = \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right)$. We claim that the coset representatives in this case are as we stated in Definition 4.2.

4.2. Commutativity.

Proposition 4.2. *Let $d, e \in (\mathbb{Z}/N\mathbb{Z})^\times$ and let p, q be primes. Then*

(i) $\langle d\rangle T_p = T_p\langle d\rangle$
(ii) $\langle d\rangle\langle e\rangle = \langle e\rangle\langle d\rangle = \langle de\rangle$
(iii) $T_pT_q = T_qT_p$.

Proof. (sketch) Let $f \in M_k(\Gamma_1(N))$.

(i) We claim that

$$\langle d\rangle T_p f(\tau) = \sum_j f|_k(\beta_j\alpha)(\tau) = \sum_j f|_k(\alpha\beta_j)(\tau) = T_p\langle d\rangle f(\tau),$$

where $\alpha \in \Gamma_0(N)$ has lower right entry congruent to d modulo N and $\{\beta_j\}_j$ are coset representatives of $\Gamma_1(N)\setminus\Gamma_1(N)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)\Gamma_1(N)$. We have equality in the middle because $\Gamma_1(N) \lhd \Gamma_0(N)$ and this implies $\cup_j\Gamma_1(N)\alpha\beta_j = \cup_j\Gamma_1(N)\beta_j\alpha$.

(ii) To show this, we use the decomposition of $M_k(\Gamma_1(N))$ as:

$$M_k(\Gamma_1(N)) = \bigoplus_{\substack{\chi \text{ - Dir char mod } N\\ \chi(-1)=(-1)^k}} M_k(\Gamma_0(N),\chi), \tag{1}$$

and the fact that

$$M_k(\Gamma_0(N),\chi) = \{f \in M_k(\Gamma_1(N) : \langle d\rangle f = \chi(d)f, \forall d \in (\mathbb{Z}/N\mathbb{Z})^\times\}$$

(iii) We again use (1) and the fact that, for $M_k(\Gamma_0(N),\chi)$, the operator T_p acts on Fourier coefficients like:

$$a_n(T_pf) = a_{np}(f) + \chi(p)p^{k-1}a_{\frac{n}{p}}(f),$$

which is Proposition 5.2.2, (ii) in [2].

□

4.3. **Extending definitions multiplicatively.**

Definition 4.4 ($\langle n\rangle$)**.** For $n \in \mathbb{N}$, let

$$\langle n\rangle := \begin{cases}\langle n(\mathrm{mod} N)\rangle, & \text{if } (n,N)=1\\ 0, & \text{otherwise.}\end{cases}$$

Note that this makes the generalised diamond operator totally multiplicative.

Definition 4.5 (T_n)**.** Set $T_1 = 1$. We already have T_p, so next we define T_{p^k} inductively by

$$T_{p^k} = T_pT_{p^{k-1}} - p^{k-1}\langle p\rangle T_{p^{k-2}}.$$

This definition is motivated by the action of T_p on Fourier coefficients of modular forms $f \in M_k(\Gamma_1(N))$:

$$a_n(T_pf) = a_{np}(f) + \mathbf{1}_N(p)p^{k-1}a_{\frac{n}{p}}(\langle p\rangle f),$$

where $\mathbf{1}_N(p) = 1$ if $p \nmid N$ and $\mathbf{1}_N(p) = 0$ otherwise. It is then automatically true that

$$T_{p^k}T_{q^l} = T_{q^l}T_{p^k}.$$

Last, but not least, if $n = \prod_i p_i^{k_i}$, define

$$T_n = \prod_i T_{p_i^{k_i}},$$

which makes the T_n commutative and multiplicative: $T_{nm} = T_n T_m$ when $(n, m) = 1$.

4.4. **Normality.** Due to the restrictions imposed by Definition 3.4 of the Petersson scalar product (namely that we need cusp forms), we restrict to the space of cusp forms:

Theorem 4.3. *Restricting $\langle p \rangle$ and T_p to the space $S_k(\Gamma_1(N))$, we have, when $p \nmid N$:*

$$\langle p \rangle^* = \langle p \rangle^{-1} (:= |_k \alpha^{-1})$$
$$T_p^* = \langle p \rangle^{-1} T_p.$$

Proof. (omitted) See §5.5 of [2]. □

Remark 1. Three remarks:

(i) This theorem extends multiplicatively to general $n \in \mathbb{N}$ which is coprime to N.
(ii) If $(n, N) > 1$, clearly $\langle n \rangle^* = 0$. The dual of T_n will contain the Atkin–Lehner involution w_N, but we will not worry about that now. Just note that:
(iii) In view of Theorem 1.1, we now know that $S_k(\Gamma_1(N))$ has an orthogonal basis of *eigenforms* (=eigenvector+modular form) for all Hecke operators $\langle n \rangle$ and T_n with $(n, N) = 1$.

5. Old and New

We can now concentrate on splitting $S_k(\Gamma_1(N))$ into oldforms and newforms. The first observation we make is that, for any $M \mid N$, we have $\Gamma_1(N) \subseteq \Gamma_1(M)$ and the reverse inclusion for modular forms: $M_k(\Gamma_1(M)) \subseteq M_k(\Gamma_1(N))$. The second observation is that, for each M and δ in $\mathbb{N}$, if $f \in M_k(\Gamma_1(M))$, then $f(\delta\tau) \in M_k(\Gamma_1(\delta M))$: say $g(\tau) = f(\delta\tau)$ and observe that, if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(dM)$, then

$$g(\tau)|_k \gamma(\tau) = (c\tau + d)^{-k} g(\gamma\tau) = \left(\frac{c}{\delta}(\delta\tau) + d\right)^{-k} f\left(\frac{a(\delta\tau) + \delta b}{\frac{c}{\delta}(\delta\tau) + d}\right)$$
$$= f|_k \begin{pmatrix} a & \delta b \\ \frac{c}{\delta} & d \end{pmatrix} (\delta\tau) = g(\tau),$$

where we have used the fact that $c \equiv 0 \bmod \delta M$ and so $\begin{pmatrix} a & \delta b \\ \frac{c}{\delta} & d \end{pmatrix} \in \Gamma_1(M)$. Thus, for every $M \mid N$ and every divisor d of N/M, $f(d\tau) \in M_k(\Gamma_1(N))$. We combine these two observations into

Definition 5.1. For $d \mid N$, define the operator

$$i_d : (S_k(\Gamma_1(N/d)))^2 \to S_k(\Gamma_1(N)),$$

which acts as:

$$i_d : (f, g) \mapsto f + g(d\tau).$$

We use this map to define

Definition 5.2 (Oldforms and Newforms)**.** Let

$$S_k(\Gamma_1(N))^{old} := \sum_{p|N} i_p(S_k(\Gamma_1(N/p)))^2,$$
$$S_k(\Gamma_1(N))^{new} := (S_k(\Gamma_1(N))^{old})^{\perp},$$

where the orthogonal complement is taken with respect to the Petersson scalar product.

We quote the following result without proof:

Theorem 5.1. *The spaces $S_k(\Gamma_1(N))^{old}$ and $S_k(\Gamma_1(N))^{new}$ are each preserved by the Hecke operators $\langle n \rangle$ and T_n, for any $n \in \mathbb{N}$. As a consequence, they each have orthogonal bases of eigenforms for this family of Hecke operators.*

6. Newforms

Definition 6.1. (Newform) We drop the "for Hecke operators" from now on and simply call eigenform any modular form which is an eigenform for all T_n and all $\langle n \rangle$ with $n \in \mathbb{N}$. An eigenform is *normalised* if $a_1(f) = 1$. A *newform* is a normalised eigenform in $S_k(\Gamma_1(N))^{new}$.

Finally, some examples:

Example 6.1. The discriminant function $\Delta(\tau)$ is a newform, since $S_{12}(\Gamma)$ is the "first" space of cusp forms, it is one-dimensional and $\tau(1) = 1$. Extending our definitions for non-cups forms allows us to say that certain Eisenstein series, when properly normalised, are also examples of newforms.

We would like to finish our exposition with the following result:

Theorem 6.1 (Multiplicity one)**.** *Let $f \in S_k(\Gamma_1(N))^{new}$ be an eigenform for the Hecke operators T_n and $\langle n \rangle$ for all n coprime to N. If $\tilde{f}$ satisfies the same conditions as f and has the same T_n-eigenvalues, then $\tilde{f} = c \cdot f$ for some constant c.*

Proof. We will use the fact that if f is an eigenform for all Hecke operators away from N, with eigenvalue λ_n for T_n, say, then

$$\lambda_n a_1(f) = a_n(f). \tag{2}$$

This follows from the formula for the action of the Hecke operators T_n on the Fourier coefficients of f and the definitions. Now, let f and $\tilde{f}$ have Fourier expansions of the form

$$f(\tau) = \sum_{n=1}^{\infty} a_n(f) q^n,$$
$$\tilde{f}(\tau) = \sum_{n=1}^{\infty} a_n(\tilde{f}) q^n.$$

Then, if we let $h(\tau) = a_1(\tilde{f})f - a_1(f)\tilde{f}$ and apply each Hecke operator T_n away from the level N, bearing in mind that f and $\tilde{f}$ have the same eigenvalues λ_n, say, we obtain:

$$T_n h(\tau) = a_1(\tilde{f})\lambda_n f - a_1(f)\lambda_n \tilde{f} = \lambda_n h(\tau),$$

i.e. $h(\tau)$ is an eigenform for all Hecke operators away from N. But we also have by construction that h is in the space of newforms and that $a_1(h) = 0$. Thus, using (2), $a_n(h) = 0$ for all n coprime to N. The missing ingredient is the Main Lemma in §5.7 of [2], which we quote without proof:

Lemma 6.2 (Main Lemma)**.** *If $f \in S_k(\Gamma_1(N))$ is such that its Fourier coefficients $a_n(f)$ vanish whenever $(n, N) = 1$, then f can be written as $\sum_{p|N} f_p(p\tau)$ with each $f_p \in S_k(\Gamma_1(N/p))$, i.e. f is an oldform by definition.*

Thus, h is an oldform and a newform at the same time, i.e. it is zero, and so $\tilde{f} = \frac{a_1(\tilde{f})}{a_1(f)} f$. □

Remark 2. The Main Lemma comes as a "converse theorem", because note that if $f \in M_k(\Gamma_1(N/p))$, then $f(p\tau)$ has a Fourier expansion of the form $\sum a_n(f)q^{np}$, i.e. oldforms of the type $\sum_{p|N} f_p(p\tau)$ have Fourier coefficients a_n that vanish when n is coprime to N. Also, Theorem 6.1 tells us that the eigenspaces of T_n are one-dimensional.

References

[1] Cohen, Henri and Strömberg, Fredrik, (in preparation). *Modular Forms: A Classical Approach.*

[2] Diamond, Fred and Shurman, Jerry, 2005.*A First Course in Modular Forms.* USA: Springer Science+Business Media, Inc.