Eisenstein series for Jacobi forms of lattice index

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Structure of the talk

1. Jacobi forms

- the arithmetic theory of scalar Jacobi forms was developed in 1985 by Eichler & Zagier
  - Eisenstein series
  - Taylor expansions
  - Hecke operators
  - relation with half-integral weight elliptic modular forms and with vector-valued modular forms (theta expansion)
  - relation with Siegel modular forms
- Jacobi forms and Algebraic Geometry (from the work of V. Gritsenko)
  - orthogonal modular forms can be obtained as lifts of Jacobi forms; the former determine Lorentzian Kac–Moody Lie (super) algebras of Borcherds type
  - Jacobi forms are solutions to the *mirror symmetry* problem for $K3$ surfaces
  - the *elliptic genus* of a Calabi–Yau manifold is a weak Jacobi form
  - and much more...

- $e_c(x) = \exp\left(\frac{2\pi i x}{c}\right)$ and $e(x) = e_1(x)$
- the *weight* of a Jacobi form will be k in $\mathbb{N}$ and the *index* $\underline{L} = (L, \beta)$:
  - $L \simeq \mathbb{Z}^{\mathrm{rk}(\underline{L})}$ is a *free, finite rank* $\mathbb{Z}$-module
  - $\beta : L \times L \to \mathbb{Z}$ is a *symmetric*, *positive-definite*, *even* $\mathbb{Z}$-bilinear form
- set $\beta(\lambda) := \frac{1}{2}\beta(\lambda, \lambda)$
- the *dual lattice* of $\underline{L}$: $L^\# := \{t \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(\lambda, t) \in \mathbb{Z}$ for all λ in $L\}$
  - the *determinant* of $\underline{L}$ is $\det(\underline{L}) := |L^\# / L|$
- the integral *Jacobi group associated to* $\underline{L}$ is $J^{\underline{L}} := \Gamma \ltimes L^2$
- $J^{\underline{L}}$ acts on $\mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C})$; given $g = (A, (\lambda, \mu))$ in $J^{\underline{L}}$, set

$$\begin{aligned} \phi|_{k,\underline{L}} g(\tau, z) := \phi\left(A\tau, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)(c\tau + d)^{-k} \\ \times e\left(\frac{-c\beta(z + \lambda\tau + \mu)}{c\tau + d} + \tau\beta(\lambda) + \beta(\lambda, z)\right) \end{aligned}$$

Definition

A function ϕ in $\mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C})$ is called a Jacobi form of weight k and index $\underline{L}$ if:

1. $\phi|_{k,\underline{L}}(A,h) = \phi$, for all (A,h) in $J^{\underline{L}}$;
2. ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{D \in \mathbb{Q}_{\leq 0}, t \in L^{\#} \\ \beta(t) - D \in \mathbb{Z}}} C(D,t) e\left((\beta(t) - D)\tau + \beta(t,z)\right).$$

- for fixed k and $\underline{L}$, denote the $\mathbb{C}$-vector space of all such functions by $J_{k,\underline{L}}$
- *Jacobi cusp forms*: $D < 0$; denote the subspace of cusp forms of weight k and index $\underline{L}$ by $S_{k,\underline{L}}$

Why the interest?

- they are an *example* of Jacobi forms

$$\vartheta_{\underline{L},y}(\tau,z) = \sum_{\substack{t \in L^{\#} \\ t \equiv y \bmod L}} e\left(\beta(t)\tau + \beta(t,z)\right)$$

- they should be perpendicular to Jacobi cusp forms with respect to a suitably defined Petersson scalar product and hence we obtain the following decomposition:

$$J_{k,\underline{L}} = S_{k,\underline{L}} \oplus J_{k,\underline{L}}^{Eis}$$

- we are interested in a theory of *newforms* with respect to Hecke operators (defined by Ajouz in 2015)

- the *isotropy set* of $\underline{L}$ is $\mathrm{Iso}(D_{\underline{L}}) := \{r \in L^\#/L : \beta(r) \in \mathbb{Z}\}$
- define $J_\infty^{\underline{L}} := \{((\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}), (0, \mu)) : n \in \mathbb{Z}, \mu \in L\}$

Definition

For every r in $\mathrm{Iso}(D_{\underline{L}})$, let $g_{\underline{L},r}(\tau, z) := e(\beta(r)\tau + \beta(r, z))$ and define the Eisenstein series of weight k and index $\underline{L}$ associated to r as

$$E_{k,\underline{L},r}(\tau, z) := \frac{1}{2} \sum_{\gamma \in J_\infty^{\underline{L}} \backslash J^{\underline{L}}} g_{\underline{L},r}|_{k,\underline{L}}\gamma(\tau, z).$$

- it is absolutely and uniformly convergent on compact subsets of $\mathfrak{H} \times (L \otimes \mathbb{C})$ for $k > \frac{\mathrm{rk}(L)}{2} + 2$
- their *twists* by primitive Dirichlet characters modulo divisors of N_r (order of r in $L^\#/L$) form a *basis of eigenforms* of $\mathrm{Span}\{E_{k,\underline{L},r} : r \in \mathrm{Iso}(D_{\underline{L}})\}$

$$E_{k,\underline{L},r,\chi}(\tau, z) := \sum_{d \in (\mathbb{Z}/\mathbb{Z}N_r)^\times} \chi(d) E_{k,\underline{L},dr}(\tau, z)$$

with eigenvalues given by *twisted divisor sums*

- consider $\underline{L}_m = (\mathbb{Z}, (x, y) \mapsto 2mxy)$, with m in $\mathbb{N}$
  - the dual is $\frac{1}{2m}\mathbb{Z}$
- then $J_{k,\underline{L}_m}$ is the space of scalar Jacobi forms $J_{k,m}$
- take $\underline{L}_m$ as the index, $r = 0$ and k even; then $E_{k,\underline{L}_m,0}$ has Fourier coefficients $C\left(0, \frac{t}{2m}\right) = 1$ for all $\frac{t}{2m}$ in $\mathbb{Z}$ and

$$C\left(\frac{t^2}{4m} - n, \frac{t}{2m}\right) = \frac{(2\pi)^{k-\frac{1}{2}} i^k}{(2m)^{\frac{1}{2}} \Gamma(k - \frac{1}{2})} \left(n - \frac{t^2}{4m}\right)^{k-\frac{3}{2}} \sum_{c \geq 1} c^{-k} \times \sum_{\substack{\lambda, d (\bmod c) \\ (d,c)=1}} e_c(md^{-1}\lambda^2 - 2mr\lambda + nd).$$

  - when $m = 1$, we have $C\left(\frac{t^2}{4} - n, \frac{t}{2}\right) = \frac{L_{t^2-4n}(2-k)}{\zeta(3-2k)}$
  - when m is square-free, we have

$$C\left(\frac{t^2}{4m} - n, \frac{t}{2m}\right) = \frac{1}{\zeta(3-2k)\sigma_{k-1}(m)} \sum_{d|(n,t,m)} d^{k-1} L_{\frac{t^2-4nm}{d^2}}(2-k)$$

Theorem (M., 2017)

For $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$ and for every r in $\mathrm{Iso}(D_{\underline{L}})$, the Eisenstein series $E_{k,\underline{L},r}$ is an element of $J_{k,\underline{L}}$ and it is orthogonal to $S_{k,\underline{L}}$ with respect to a suitably defined Petersson scalar product. It has the following Fourier expansion:

$$E_{k,\underline{L},r}(\tau,z) = \frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z) + (-1)^k \vartheta_{\underline{L},-r}(\tau,z)\right) + \sum_{\substack{D \in \mathbb{Q}_{<0}, t \in L^\# \\ \beta(t)-D \in \mathbb{Z}}} C_{k,\underline{L},r}(D,t) e\left((\beta(t)-D)\tau + \beta(t,z)\right),$$

where

$$C_{k,\underline{L},r}(D,t) = \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k}{2\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} (-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \times \sum_{c \geq 1} c^{-k} \left(H_{\underline{L},c}(r,D,t) + (-1)^k H_{\underline{L},c}(-r,D,t)\right)$$

and $H_{\underline{L},c}(r,D,t)$ is the lattice sum

$$\sum_{\lambda \in L/cL, d \in (\mathbb{Z}/c\mathbb{Z})^\times} e_c\left(\beta(\lambda + r) d^{-1} + (\beta(t) - D)d + \beta(t, \lambda + r)\right).$$

$$E_{k,\underline{L},r}(\tau,z) = \frac{1}{2} \sum_{\gamma \in J_\infty^{\underline{L}} \backslash J^{\underline{L}}} g_{\underline{L},r}|_{k,\underline{L}} \gamma(\tau,z) \qquad \text{(Reminder)}$$

- invariance under $|_{k,\underline{L}}$ action of $J^{\underline{L}}$ follows by construction
- the given Fourier expansion completes the modularity argument
- for orthogonality to cusp forms, use unfolding technique:

$$\begin{aligned}\langle \phi, E_{k,\underline{L},r} \rangle &:= \int_{\mathfrak{F}_{J^{\underline{L}}}} \phi(\tau,z) \sum_{\gamma \in J_\infty^{\underline{L}} \backslash J^{\underline{L}}} \overline{g_r|_{k,\underline{L}}\gamma(\tau,z)} \Im(\tau)^k e^{-\frac{4\pi\beta(\Im(z))}{\Im(\tau)}} dV_{\underline{L}}(\tau,z) \\ &= \int_{\mathfrak{F}_{J_\infty^{\underline{L}}}} \phi(\tau,z) \overline{g_r(\tau,z)} \Im(\tau)^k e^{-\frac{4\pi\beta(\Im(z))}{\Im(\tau)}} dV_{\underline{L}}(\tau,z)\end{aligned}$$

- choose a convenient fundamental domain
- insert Fourier expansion of ϕ and use orthogonality relations for exponential function, which imply that the integral in $\Re(z)$ vanishes

$$E_{k,\underline{L},r}(\tau,z) = \frac{1}{2} \sum_{\gamma \in J_\infty^{\underline{L}} \backslash J^{\underline{L}}} g_{\underline{L},r}|_{k,\underline{L}}\gamma(\tau,z)$$ (Reminder)

- choose a convenient set of coset representatives for $J_\infty^{\underline{L}} \setminus J^{\underline{L}}$
- write each γ as $(A, (a\lambda, b\lambda))$ and separate contributions coming from terms with $c = 0$ and $c \neq 0$
  - $c = 0$ gives the singular term

$$\sum_{\substack{t \in L^\# \\ t \equiv r \bmod L}} e\left(\beta(t)\tau\right)\left(e\left(\beta(t,z)\right) + (-1)^k e\left(\beta(-t,z)\right)\right)$$

  - terms with $c < 0$ give same contribution as those with $c > 0$, multiplied by $(-1)^k$ and with z replaced by $-z$
  - contribution coming from $c > 0$ is:

$$\sum_{c>0} c^{-k} \sum_{\lambda(c), d(c)^\times} e_c\left(\beta(\lambda + r)d^{-1}\right) \mathcal{F}\left(\tau + \frac{d}{c}, z - \frac{1}{c}\lambda\right),$$

  where $\mathcal{F}(\tau, z)$ has period $\mathbb{Z}$ in τ and period L in z.

Theorem (M., 2017)

For every r in $\mathrm{Iso}(D_{\underline{L}})$, the Eisenstein series $E_{k,\underline{L},r}$ is an element of $J_{k,\underline{L}}$ and it is orthogonal to $S_{k,\underline{L}}$ with respect to a suitably defined Petersson scalar product. It has the following Fourier expansion:

$$E_{k,\underline{L},r}(\tau,z) = \frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z) + (-1)^k \vartheta_{\underline{L},-r}(\tau,z)\right) + \sum_{\substack{D\in\mathbb{Q}_{<0},t\in L^{\#} \\ \beta(t)-D\in\mathbb{Z}}} C_{k,\underline{L},r}(D,t) e\left((\beta(t)-D)\tau + \beta(t,z)\right),$$

where

$$C_{k,\underline{L},r}(D,t) = \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k}{2\det(\underline{L})^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}(-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \times \sum_{c\geq 1} c^{-k}\left(H_{\underline{L},c}(r,D,t) + (-1)^k H_{\underline{L},c}(-r,D,t)\right)$$

and $H_{\underline{L},c}(r,D,t)$ is the lattice sum

$$\sum_{\lambda\in L/cL, d\in(\mathbb{Z}/c\mathbb{Z})^{\times}} e_c\left(\beta(\lambda+r)d^{-1} + (\beta(t)-D)d + \beta(t,\lambda+r)\right).$$

- suppose that rk($\underline{L}$) is even
- define $\Delta(\underline{L}) := (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \det(\underline{L})$ and write $\Delta(\underline{L}) = \mathfrak{d}\mathfrak{f}^2$, with $\mathfrak{d}$ the discriminant of $\mathbb{Q}(\sqrt{\Delta(\underline{L})})$; set $\chi_d(\cdot) := \left(\frac{d}{\cdot}\right)$

Theorem (M., 2017)

When k is odd, $E_0 \equiv 0$. When k and rk($\underline{L}$) are even, E_0 has the Fourier expansion

$$E_0(\tau, z) = \vartheta_{\underline{L},0}(\tau, z) + \sum_{\substack{(D,t)\in \mathit{supp}(\underline{L}) \\ D<0}} C_0(D,t) e\left((\beta(t) - D)\tau + \beta(t,z)\right)$$

and

$$C_0(D,t) = \frac{2(-1)^{\lceil \frac{\mathit{rk}(\underline{L})}{4} \rceil} (-D|\mathfrak{d}|)^{k-\frac{\mathit{rk}(\underline{L})}{2}-1}}{\mathfrak{f} L\left(1-k+\frac{\mathit{rk}(\underline{L})}{2}, \chi_{\mathfrak{d}}\right) \sum_{d|\mathfrak{f}} \frac{\mu(d)}{d^{k-\frac{\mathit{rk}(\underline{L})}{2}}} \chi_{\mathfrak{d}}(d) \sigma_{1-2k+\mathit{rk}(\underline{L})}\left(\frac{\mathfrak{f}}{d}\right)} \times \prod_{p | 2DN_t^2 \det(\underline{L})} \frac{L_p(k-1)}{1-\chi_{\Delta(\underline{L})}(p) p^{\frac{\mathit{rk}(\underline{L})}{2}-k}}.$$

- define the representation numbers

$$R(b) := \#\{\lambda \in L/bL : \beta(\lambda + t) - D \equiv 0 \bmod b\}$$

- set $r = 0$ in the formula for $C_r(D,t)$, introduce the Möbius function to remove the coprimality conditions in $H_{\underline{L},c}(0,D,t)$ and obtain

$$C_{k,\underline{L},0}(D,t) = \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{2\det(\underline{L})^{-\frac{1}{2}}\Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)\zeta(k - \mathrm{rk}(\underline{L}))}(1+(-1)^k)\sum_{b\geq 1}\frac{R(b)}{b^{k-1}}$$

- reminder:

$$\text{singular-term}(E_{k,\underline{L},0}) = \frac{1+(-1)^k}{2}\vartheta_{\underline{L},0}$$

- the $R(b)$'s are multiplicative functions of b; the Dirichlet series $L(s) := \sum_{b\geq 1} R(b)b^{-s}$ arises in Brunier & Kuss (2001)
- it converges for $\Re(s) > \mathrm{rk}(\underline{L})$ and that it can be continued meromorphically to $\Re(s) > \frac{\mathrm{rk}(\underline{L})}{2} + 1$, with a simple pole at $s = \mathrm{rk}(\underline{L})$

- define $w_p = 1 + 2\mathrm{ord}_p(2N_t D)$ and the local Euler factor

$$L_p(s) := p^{-w_p s} R(p^{w_p}) + \left(1 - p^{-(s - \mathsf{rk}(\underline{L}) + 1)}\right) \sum_{l=0}^{w_p - 1} p^{-ls} R(p^l)$$

- using results of Siegel (1935) on representation numbers of quadratic forms modulo prime powers, we obtain

$$L(s) = \frac{\zeta(s - \mathsf{rk}(\underline{L}) + 1)}{L\left(s - \frac{\mathsf{rk}(\underline{L})}{2} + 1, \chi_{\Delta(\underline{L})}\right)} \prod_{p \mid 2DN_t^2 \det(\underline{L})} \frac{L_p(s)}{1 - \chi_{\Delta(\underline{L})}(p) p^{-(s - \frac{\mathsf{rk}(\underline{L})}{2} + 1)}}$$

- functional equations, Gauss sums, the duplication formula and Euler's reflection formula for the Gamma function, values of the Riemann zeta function at positive even integers complete the proof

- we have obtained:

$$C_0(D,t) = \frac{2(-1)^{\lceil \frac{\mathsf{rk}(\underline{L})}{4} \rceil}(-D|\mathfrak{d}|)^{k-\frac{\mathsf{rk}(\underline{L})}{2}-1}}{\mathfrak{f} L\left(1-k+\frac{\mathsf{rk}(\underline{L})}{2}, \chi_{\mathfrak{d}}\right) \sum_{d|\mathfrak{f}} \frac{\mu(d)}{d^{k-\frac{\mathsf{rk}(\underline{L})}{2}}} \chi_{\mathfrak{d}}(d) \sigma_{1-2k+\mathsf{rk}(\underline{L})}\left(\frac{\mathfrak{f}}{d}\right)} \times \prod_{p|2DN_t^2 \det(\underline{L})} \frac{L_p(k-1)}{1-\chi_{\Delta(\underline{L})}(p) p^{\frac{\mathsf{rk}(\underline{L})}{2}-k}}$$

Corollary

The Fourier coefficients of E_0 are rational numbers.

- use results of Zagier (1981)

$$L(-n,\chi) = -\frac{M^n}{n+1} \sum_{l=1}^{M} \chi(l) B_{n+1}\left(\frac{l}{M}\right)$$

- can we say something about the 'bad' Euler factors?
- Cowan, Katz and White (2017): set $\mathfrak{D} := \prod_{\substack{p \mid N_t^2 D \\ \gcd(p, 2\det(\underline{L}))=1}} p^{v_p(N_t^2 D)}$

$$\prod_{\substack{p \mid N_t^2 D \\ p \nmid 2\det(\underline{L})}} \frac{L_p(k-1)}{1-\chi_{\underline{L}}(p)p^{-\left(k-\frac{\mathsf{rk}(\underline{L})}{2}\right)}} = \chi_{\underline{L}}(\mathfrak{D})\mathfrak{D}^{-\left(k-\frac{\mathsf{rk}(\underline{L})}{2}-1\right)} \sum_{d \mid \mathfrak{D}} \chi_{\underline{L}}(d) d^{k-\frac{\mathsf{rk}(\underline{L})}{2}-1}$$

- the formulas in CKW were implemented in Sage by B. Williams: `https://math.berkeley.edu/~btw/local-L-functions.sagews`

Example

If $\underline{L}$ is unimodular ($L^{\#} = L$), then

$$C_0(D,t) = \frac{\mathit{rk}(\underline{L}) - 2k}{B_{k-\frac{\mathit{rk}(\underline{L})}{2}}} \sigma_{k-\frac{\mathit{rk}(\underline{L})}{2}-1}(D).$$

This was also shown by Woitalla (2018).

- let $x \in L^{\#}/L$ and define the following Schrödinger representation $\sigma_x : \mathbb{Z}^3 \to \mathrm{Span}_{\mathbb{C}}\{\vartheta_{\underline{L},y} : y \in L^{\#}/L\}$:

$$\sigma_x(\lambda, \mu, \nu)\vartheta_{\underline{L},y} := e\left(\mu\beta(x,y) + (\nu - \lambda\mu)\beta(x)\right)\vartheta_{\underline{L},y-\lambda x}$$

- note that $\mathbb{Z}^3$ has group law

$$(\lambda, \mu, t)(\lambda', \mu', t') = (\lambda + \lambda', \mu + \mu', t + t' + \lambda\mu' - \mu\lambda').$$

- every Jacobi form has a *theta expansion*:

$$\phi(\tau, z) = \sum_{y \in L^{\#}/L} h_{\phi,y}(\tau)\vartheta_{\underline{L},y}(\tau, z),$$

where

$$h_{\phi,y}(\tau) = \sum_{\substack{D \in \mathbb{Q} \\ (D,y) \in \mathrm{supp}(\underline{L})}} C(D,y)q^{-D}$$

Definition

Define the averaging operator at x in the following way:

$$\mathsf{Av}_x \phi(\tau, z) := \frac{1}{N_x^2} \sum_{(\lambda,\mu)\in(\mathbb{Z}/\mathbb{Z}N_x^2)^2} \sigma_x^*(\lambda, \mu, 0)\phi(\tau, z).$$

- defined for vector-valued modular forms by Williams (2018)
- σ_x is unitary and Av_x maps $J_{k,\underline{L}}$ to $J_{k,\underline{L}}$

Proposition (M., 2017)

Suppose that k is even and let r be an element of $\mathrm{Iso}(D_{\underline{L}})$. Then:

$$\sum_{\lambda\in\mathbb{Z}/\mathbb{Z}N_r} E_{k,\underline{L},\lambda r}(\tau, z) = \sum_{\substack{t\in L^{\#}/L \\ \beta(r,t)\in\mathbb{Z}}} \left(\sum_{\lambda\in\mathbb{Z}/\mathbb{Z}N_r} h_{E_{k,\underline{L},0},t+\lambda r}(\tau) \right) \vartheta_{\underline{L},t}(\tau, z).$$

- for any r in $L^{\#}/L$,

$$\mathsf{Av}_r E_{k,\underline{L},0}(\tau,z) = \sum_{\substack{\lambda\in\mathbb{Z}/\mathbb{Z}N_r^2 \\ \lambda\beta(r)\in\mathbb{Z}}} E_{k,\underline{L},\lambda r}(\tau,z)$$

- insert definition of Av_r and theta expansion of $E_{k,\underline{L},0}$ on left-hand side and expand:

$$\mathsf{Av}_r E_{k,\underline{L},0}(\tau,z) = N_r \sum_{\substack{t\in L^{\#}/L \\ \beta(r,t)\in\mathbb{Z}}} h_{E_{k,\underline{L},0},t}(\tau) \sum_{\lambda\in\mathbb{Z}/\mathbb{Z}N_r} \vartheta_{\underline{L},t-\lambda r}$$

- if $\beta(r)\in\mathbb{Z}$, then trivially

$$\sum_{\lambda\in\mathbb{Z}/\mathbb{Z}N_r^2} E_{k,\underline{L},\lambda r}(\tau,z) = N_r \sum_{\lambda\in\mathbb{Z}/\mathbb{Z}N_r} E_{k,\underline{L},\lambda r}(\tau,z),$$

- with this formula, we can compute Fourier coefficients of $E_{k,\underline{L},r}$ for any r of order $2, 3, 4$ or 6

Example

Suppose that r in has order 2. Then:

$$E_{k,\underline{L},0}(\tau,z) + E_{k,\underline{L},r}(\tau,z) = \sum_{\substack{t\in L^{\#}/L \\ \beta(r,t)\in\mathbb{Z}}} \left(h_{E_{k,\underline{L},0},t} + h_{E_{k,\underline{L},0},t+r}\right)(\tau)\vartheta_{\underline{L},t}(\tau,z)$$

and hence

$$C_{k,\underline{L},r}(D,t) = \begin{cases} C_{k,\underline{L},0}(D,t+r), & \text{if } \beta(r,t)\in\mathbb{Z} \\ -C_{k,\underline{L},0}(D,t), & \text{otherwise} \end{cases}$$

- Poincaré series:

$$C_{k,\underline{L},F,r}(D,t) := \delta_L(F,r,D,t) + (-1)^k \delta_L(F,-r,D,t) + \frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}}$$
$$\times \left(\frac{D}{F}\right)^{\frac{k}{2}-\frac{\mathsf{rk}(\underline{L})}{4}-\frac{1}{2}} \sum_{c\geq 1} J_{k-\frac{\mathsf{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DF)^{\frac{1}{2}}}{c}\right) c^{-\frac{\mathsf{rk}(\underline{L})}{2}-1}$$
$$\times \left(H_{\underline{L},c}(F,r,D,t) + (-1)^k H_{\underline{L},c}(F,-r,D,t)\right)$$

- Schwagenscheidt (2018): Fourier coefficients of $E_{D_{\underline{L}},r,\chi}$
- Ran and Skoruppa (in progress): Fourier coefficients of $E_{k,\underline{L},r,\chi}$

Thank you!