GEOMETRIC MODULAR FORMS
UON MATHEMATICS
STUDY GROUP ON 'MODULI SPACES'

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1. Lecture 1

1.1. **Reminders.** Remember the following:

Definition 1.1 (Elliptic curve)**.** Let R be a field and S and R-scheme. An elliptic curve over S is a proper, smooth morphism $p : E \to S$ such that the geometric fibers are connected curves of genus 1, together with a section $e : S \to E$.

We have seen that if $R = \mathbb{Z}[\frac{1}{6}]$ (or any field of characteristic $\neq 2, 3$), then the set:

$$\mathcal{W}(S) = \left\{ \begin{matrix} (E, \omega) : E \text{ is an e.c}/S, \\ \omega \text{ is a non-zero invariant differential on } E \end{matrix} \right\}_{/\simeq},$$

where $(E, \omega) \simeq (E', \omega') \iff \phi : E \to E'$ is an S-isomorphism such that $\phi^*(\omega') = \omega$, is representable by

$$W = \mathrm{Spec}(R[A, B][\frac{1}{\Delta}]),$$

where $\Delta = -16(4A^3 + 27B^2)$. If if $R = k$, with char. $k \neq 2, 3$ and we look at $\mathcal{W}(K)$ for some field extension K/k, then given an elliptic curve (E, ω), we can obtain it from the universal elliptic curve over W

$$(\mathbb{E}, \omega) : y^2 = x^3 + Ax + B, \quad \omega = \frac{dx}{2y},$$

by specializing A and B.

1.2. **Geometric modular forms.**

Definition 1.2 (Modular form of level 1)**.** Let $k \in \mathbb{Z}$, R_0 a commutative ring with unity and R and R_0-algebra. A modular form of weight k and level 1 is a rule f

$$(E/R, \omega) \mapsto f(E/R, \omega) \in R,$$

where E/R is an e.c./R and ω is a basis of $\Omega^1(E/R)$, such that:

(*i*) f only depends on the isomorphism class of (E, ω).

(*ii*) $f(E/R, \lambda\omega) = \lambda^{-k} f(E/R, \omega)$, for any $\lambda \in R^\times$.
(*iii*) f commutes with arbitrary base change, i.e. if $\psi : R \to R'$ is and R_0-algebra homomorphism which sends E to $E' = E \times_{\mathrm{Spec}(R)} \mathrm{Spec}(R')$ (with $p : E' \to E$), then if $\omega' := p^*\omega$, we have $f(E'/R', \omega') = \phi(f(E/R, \omega))$.

Note that we could have instead defined f as a rule $E/S \to f(E/S)$, a section of $\underline{\omega}_{E/S}^{\otimes k}$, where $\underline{\omega}_{E/S}^{\otimes k} = p_*(\omega^1_{E/S})$ is an invertible sheaf on S. Then, if S =Spec(R) and $\underline{\omega}_{E/R}$ is a free $\mathbb{Z}$-module with basis ω, we would have $f(E/\mathrm{Spec}R) = f(E/R, \omega)\omega^{\otimes k}$.

They form an R_0-module $M(R_0, 1, k)$.

Example 1.1. We have seen that every e.c./$\mathbb{C}$ is isomorphic to $E_\tau = \mathbb{C}/\mathbb{Z}\tau \oplus \mathbb{Z}$, for some $\tau \in \mathfrak{H}$. Then, $(E_\tau, \omega) = (E_{\tau'}, \omega') \iff \tau' = \gamma\tau$ for some $\gamma \in \Gamma$ (which maps $z \mapsto z' = \frac{z}{c\tau+d}$) and if $\omega = dz$, then $\omega' = (c\tau + d)dz'$ (where (c, d) are the lower entries of γ). If we take a geometric modular form $f(E_\tau, dz)$ and we define $g(\tau) = f(E_\tau, dz)$, then, for any $\gamma \in \Gamma$,

$$\begin{aligned} g(\gamma\tau) &= f(E_{\gamma\tau}, dz') = f(E_\tau, (c\tau + d)^{-1}dz) \\ &= (c\tau + d)^k g(\tau), \end{aligned}$$

so we obtain a modular form in the classical sense, provided we have the right holomorphicity conditions.

In order to talk about level N modular forms, we need to define:

Definition 1.3 (Level N structure)**.** If E/S is an e.c., a choice of isomorphism of group schemes $\alpha : (\mathbb{Z}/N\mathbb{Z})^2 \to E[N]$ is called a level N structure for E/S.

Definition 1.4 (Modular form of level N)**.** If R_0 is a $\mathbb{Z}[\frac{1}{N}]$-algebra, then a modular form of weight k and level N is a rule f

$$(E, \alpha, \omega) \mapsto f(E, \alpha, \omega),$$

where everything is as before and α is a level N structure on E, such that

(*i*) -.
(*ii*) $f(E/R, \alpha, \lambda\omega) = \lambda^{-k} f(E/R, \omega, \alpha)$, for all $\lambda \in R^\times$.
(*iii*) -.

As before, they form an R_0-algebra $M(R_0, N, k)$.

1.3. q**-expansions.** Remember the following:

Definition 1.5 (Level N Tate curve)**.** It is the elliptic curve $\mathrm{Tate}(q^N)/\mathbb{Z}((q))\otimes_{\mathbb{Z}} R_0$ given by the equation:

$$y^2 + xy = x^3 + a_4(q^N)x + a_6(q^N),$$

together with the canonical differential $\omega_{can} = \frac{dx}{x+2y}$.

Note that $\mathrm{Tate}(q^N)[N] = \{\zeta_N^i q^j : 0 \le i, j \le N-1\}$, for some primitive N-th root of unity ζ_N. Let α be a level N-structure of $\mathrm{Tate}(q^N)$ and let f be a modular form of weight k and level N over R_0, a $\mathbb{Z}[\frac{1}{N}, \zeta_N]$-algebra. Then:

Definition 1.6 (q-expansion)**.** The q-expansion of f at α is

$$(*) \quad f(\mathrm{Tate}(q^N), \alpha, \omega_{can}) \in \mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0.$$

Note that this automatically ensures meromorphicity at ∞.

Definition 1.7. We say that f is holomorphic at ∞ if $(*) \in \mathbb{Z}[[q]]\otimes_{\mathbb{Z}} R_0$ for all α and that f is a cusp form if $(*) \in q\mathbb{Z}[[q]] \otimes_{\mathbb{Z}} R_0$ for all α.

Remember that, if $R = \mathbb{Z}[\frac{1}{N}]$ and S is an R-scheme, then

$$\mathcal{M}_N(S) = \{(E, \alpha)\}_{/\simeq}$$

has a coarse moduli space (which is fine if $N \ge 3$), given by the smooth affine curve $Y(N)(R)$. Over $\mathbb{C}$,

$$Y(N)(\mathbb{C}) = \coprod_{\substack{\zeta \in \mu_N \\ \zeta \text{ primitive}}} Y(N)_\zeta(\mathbb{C}),$$

where each $Y(N)_\zeta(\mathbb{C}) = \{(E, \alpha) : \alpha = (e_1, e_2) \text{ with } \langle e_1, e_2\rangle_{\mathrm{Weil}} = \zeta\}_{/\simeq} \simeq \Gamma(N) \setminus \mathfrak{H}$.

We have the following:

Theorem 1.1. *Let R_0 be a $\mathbb{Z}[\frac{1}{N}, \zeta_N]$-algebra and f a modular form which is holomorphic at ∞. If, for any primitive N-th root of unity ζ, there exists a level N-structure $\alpha_\zeta = (e_1, e_2)$ on Tate(q^N) with $\langle e_1, e_2\rangle_{Weil} = \zeta$, such that the q-expansion of f at $\alpha_\zeta = 0$, then $f \equiv 0$.*

Proof. See Katz's original '72 paper. □

The main consequence is:

Corollary 1.2 (q-expansion principle)**.** *Let f be a modular form which is holomorphic at ∞ and has coefficients in some R_0-module K. If, on each of the $\phi(N)$ connected components of $Y(N)(R)$, there is at least one cusp at which the q-coefficients of f lie in some R_0-submodule of K, then f is a modular form with coefficients in that submodule.*

Proof. Short exact sequences. □

Example 1.2. We have seen (many times) the Weierstrass $\wp$-function, which satisfies the elliptic equation: $\wp'^2 = 4\wp^3 - g_2(L)\wp - g_3(L)$. We have $g_2(L) = 60G_4(L)$ and $g_3(L) = 140G_6(L)$, where

$$G_k(L) = \sum_{\beta \in L} \beta^{-k}.$$

Note that we could have defined $G_k(\tau)$ or $G_k(E, \omega)$ because the spaces they lie in are isomorphic. If we define

$$E_k(\cdot) = \frac{1}{2\zeta(k)} G_k(\cdot),$$

then

$$E_k(q) = 1 - \frac{2k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in \mathbb{Q}[[q]].$$

Furthermore, E_4 and $E_6 \in \mathbb{Z}[[q]]$ and $\Delta = \frac{E_4^3 - E_6^2}{1728}$ and $j = \frac{E_4^3}{\Delta}$.

It is then easy to see that each $E_k \in M(R_0, 1, k)$.

2. Lecture 2

2.1. Reminders.

Definition 2.1 (Modular form of level N)**.** If R_0 is a $\mathbb{Z}[\frac{1}{N}]$-algebra and R is an R_0-algebra, then a modular form of weight k and level N on R_0 is a rule f

$$(E, \alpha, \omega) \mapsto f(E, \alpha, \omega),$$

where everything is as before and α is a level N structure on E, such that

(*i*) f is defined on isomorphism classes.
(*ii*) $f(E/R, \alpha, \lambda\omega) = \lambda^{-k} f(E/R, \omega, \alpha)$, for all $\lambda \in R^\times$.
(*iii*) f is invariant under base change.

They form an R_0-algebra $M(R_0, N, k)$.

Definition 2.2 (Level N Tate curve)**.** It is the elliptic curve $\mathrm{Tate}(q^N)/\mathbb{Z}((q)) \otimes_{\mathbb{Z}} R_0$ given by the equation:

$$y^2 + xy = x^3 + a_4(q^N)x + a_6(q^N),$$

together with the canonical differential $\omega_{can} = \frac{dx}{x+2y}$.

The Tate curve gives us the q-expansion of a modular form.

Definition 2.3 (Eisenstein series)**.** We define the weight k Eisenstein series:

$$E_k(q) = 1 - \frac{2k}{B_{2k}} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \in \mathbb{Q}[[q]].$$

Then, $E_k \in M(R_0, 1, k)$ Furthermore, E_4 and $E_6 \in \mathbb{Z}[[q]]$.

3. p-adic modular forms

Let $\mathbb{Q} \cap \mathbb{Z}_p = \mathbb{Z}_{(p)}$. Then, if $p \geq 5$, for $k = p-1$ we have

$$v_p\left(\frac{-2(p-1)}{B_{p-1}}\right) = 1,$$

and consequently $E_{p-1}(q) \in \mathbb{Z}_{(p)}[[q]]$. Thus, it makes sense to reduce the coefficients modulo p and we get a modular form in $M(\mathbb{F}_p, 1, p-1)$ with q-expansion $E_{p-1}(q) = 1$.

Let E/R be an e.c., where R is an $\mathbb{F}_p$-algebra (i.e. $p = 0$ in R). Consider the absolute Frobenius map defined on the sheaf of holomorphic functions on E:

$$F_{abs} : \mathcal{O}_E \to \mathcal{O}_E, \quad f \mapsto f^p.$$

Let $\omega \in \Omega^1_{E/R}$ and let $\eta \in H^1(E, \mathcal{O}_E)$ be its dual. We have the following:

Definition 3.1 (The Hasse invariant)**.** The Hasse invariant is the map $A : (E/R, \omega) \mapsto A(E/R, \omega)$ given by the equation

$$F^*_{abs}(\eta) = A(E/R, \omega)\eta.$$

The following holds:

Proposition 3.1. *The Hasse invariant is an element of* $M(\mathbb{F}_p, 1, p-1)$.

Proof. Given an $\omega \in \Omega^1_{E/R}$, if we make the substitution $\omega \mapsto \lambda\omega$ for some $\lambda \in R^\times$, then we will have $\eta \mapsto \lambda^{-1}\eta$. Thus,

$$F^*_{abs}(\lambda^{-1}\eta) = \lambda^{-p} F^*_{abs}(\eta) = \lambda^{-p} A(E, \omega)\eta = A(E, \lambda\omega)\lambda^{-1}\eta.$$

Thus, $A(E, \lambda\omega) = \lambda^{-(p-1)} A(E, \omega)$. Furthermore, it can be shown that $A(\mathrm{Tate}(q), \omega_{can}) = 1$ and so A is holomorphic. □

By the q-expansion principle, $A = E_{p-1} \bmod p$ for $p \geq 5$ (this was shown by Deligne). For $p = 2, 3$ it is not possible to lift A to a holomorphic modular form on $\mathbb{Z}_{(p)}$, but this can be fixed by adding some very specific level structure.

We want a p-adic theory of modular forms that strongly identifies a modular form with its q-expansion so that what 'looks' invertible, as in $E_{p-1} \bmod p$, is invertible. The Hasse invariant $A(E/R, \omega) = 0$ if and only if E is supersingular. We want to somehow 'throw away' elliptic curves which are supersingular or have supersingular reduction:

Definition 3.2 (p-adic modular form)**.** Let R be an $\mathbb{F}_p$-algebra for $p \geq 5$. Then a p-adic modular form of weight k and level N is a rule f

$$(E/R, \alpha, Y, \omega) \mapsto f(E/R, \alpha, Y, \omega) \in R,$$

where $Y \in R$ is such that $YE_{p-1}(E/R, \omega) = 1$ and such that

(i) -.

(ii) $f(E/R, \alpha, \lambda^{p-1}Y, \lambda\omega) = \lambda^{-k} f(E/R, \alpha, Y, \omega)$, for any $\lambda \in R^{\times}$.

We have the following:

Theorem 3.2 (Swinnerton-Dyer)**.** *Let* $M(\mathbb{F}_p) = \sum_{k \geq 0} M(\mathbb{F}_p, 1, k)$. *Then*

$$M(\mathbb{F}_p) \simeq \mathbb{F}_p[E_4, E_6]/(A(E_4, E_6) - 1).$$

Remark 1.

(i) Not a direct sum because modular forms of different weight may have same mod$p$ q-expansion.

(ii) $M(\mathbb{C}) = \bigoplus_k M_k(\mathbb{C}) = \mathbb{C}[E_4, E_6]$.

(iii) Proof uses the commutative diagram:

$$\begin{tikzcd} \mathbb{Z}_{(p)}[E_4, E_6] \arrow[r, "\simeq"] \arrow[d, two heads] & M(\mathbb{Z}_{(p)}) \arrow[d, two heads] \\ \mathbb{F}_p[E_4, E_6] \arrow[r, two heads, "\psi"] & M(\mathbb{F}_p) \end{tikzcd}$$

Introduce derivation $\theta = q\frac{d}{dq}$ (on $\mathbb{C}[q]$) and then $\partial = 12\theta - kE_2$:

$$E_2 = 1 - 24 \sum_{n \geq 1} \sigma_1(n) q^n.$$

Then, ∂ is a derivation on $\mathbb{Z}_{(p)}[q]$ and then on $\mathbb{F}_p[q]$.

References

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