Level raising operators and Jacobi–Eisenstein series

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Structure of the talk

1. Setting
2. Motivation
3. Results

- the *weight* of a Jacobi form will be k in $\mathbb{N}$ and the *index* $\underline{L} = (L, \beta)$, where:
  - L is a *finite rank* $\mathbb{Z}$-module
  - $\beta : L \times L \to \mathbb{Z}$ is a $\mathbb{Z}$-bilinear form which is:
    - *symmetric*: $\beta(\lambda, \mu) = \beta(\lambda, \mu)$
    - *positive-definite*: $\beta(\lambda, \lambda) > 0$ unless $\lambda = 0$
    - *even*: $\beta(\lambda, \lambda) \in 2\mathbb{Z}$
- set $\beta(\lambda) := \frac{1}{2}\beta(\lambda, \lambda)$
- the *dual* of $\underline{L}$: $L^{\#} := \{t \in L \otimes \mathbb{Q} : \beta(\lambda, t) \in \mathbb{Z}, \text{ for all } \lambda \text{ in } L\}$
- the *support* of $\underline{L}$:

$$\mathrm{supp}(\underline{L}) := \{(D, t) : D \in \mathbb{Q}_{\leq 0}, t \in L^{\#} \text{ and } D \equiv \beta(t) \bmod \mathbb{Z}\}$$

- the *determinant* of $\underline{L}$: $\det(\underline{L}) := |L^{\#}/L|$
- the *level* of $\underline{L}$: $\mathrm{lev}(\underline{L}) := \min\{N \in \mathbb{N} : N\beta(t) \in \mathbb{Z} \text{ for all } t \text{ in } L^{\#}\}$

Definition (The Jacobi group associated to $\underline{L}$)

Define $J_{\underline{L}}(\mathbb{Z})$ to be the semi-direct product $\Gamma \ltimes L^2$.

- $J_{\underline{L}}(\mathbb{Z})$ acts on $\mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C})$:

$$\phi|_{k,\underline{L}}\,(A,(\lambda,y))\,(\tau,z) := \phi\left(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-k}$$
$$\times\, e\left(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,z)\right)$$

Definition (Jacobi form of lattice index)

A function ϕ in $\mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C})$ is called a Jacobi form of weight k and index $\underline{L}$ if:

1. $\phi|_{k,\underline{L}}(A,h) = \phi$, for all (A,h) in $J_{\underline{L}}(\mathbb{Z})$;
2. ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{(D,t)\in\ \mathrm{supp}(\underline{L})} C(D,t)e\left((\beta(t)-D)\tau + \beta(t,z)\right).$$

- for fixed k and $\underline{L}$, denote the $\mathbb{C}$-vector space of all such functions by $J_{k,\underline{L}}$
- as a consequence of results of Boylan (2015), $J_{k,\underline{L}} = \{0\}$ if $k < \frac{\mathrm{rk}(\underline{L})}{2}$

- define $\mathrm{Iso}(\underline{L}) := \{r \in L^{\#}/L : \beta(r) \in \mathbb{Z}\}$
- define $J_{\underline{L}}(\mathbb{Z})_\infty := \{((\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}), (0, \mu)) : n \in \mathbb{Z}, \mu \in L\}$

Definition (Jacobi-Eisenstein series)

For every r in $\mathrm{Iso}(\underline{L})$, let $g_{\underline{L},r}(\tau, z) := e(\beta(r)\tau + \beta(r, z))$ and define the Eisenstein series of weight k and index $\underline{L}$ associated to r as

$$E_{k,\underline{L},r}(\tau, z) := \frac{1}{2} \sum_{\gamma \in J_{\underline{L}}(\mathbb{Z})_\infty \backslash J_{\underline{L}}(\mathbb{Z})} g_{\underline{L},r}|_{k,\underline{L}}\gamma(\tau, z).$$

- defined by Ajouz (2015); it is absolutely and uniformly convergent on compact subsets of $\mathfrak{H} \times (L \otimes \mathbb{C})$ for $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$
- *Jacobi cusp forms* have the following type of Fourier expansion:

$$\phi(\tau, z) = \sum_{\substack{(D,t) \in \ \mathrm{supp}(\underline{L}) \\ D<0}} C(D, t) e\left((\beta(t) - D)\tau + \beta(t, z)\right)$$

- denote the subspace of cusp forms of weight k and index $\underline{L}$ by $S_{k,\underline{L}}$

Theorem (M., 2017)

The Eisenstein series $E_{k,\underline{L},r}$ is an element of $J_{k,\underline{L}}$ and it is orthogonal to $S_{k,\underline{L}}$ with respect to a suitably defined Petersson scalar product. It has the following Fourier expansion:

$$E_{k,\underline{L},r}(\tau,z) = \frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z) + (-1)^k \vartheta_{\underline{L},-r}(\tau,z)\right) + \sum_{\substack{(D,t)\in \mathrm{supp}(\underline{L}) \\ D<0}} C_{k,\underline{L},r}(D,t) e\left((\beta(t)-D)\tau + \beta(t,z)\right),$$

where

$$C_{k,\underline{L},r}(D,t) = \frac{(2\pi)^{k-\frac{\mathbf{rk}(\underline{L})}{2}} i^k}{2\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathbf{rk}(\underline{L})}{2}\right)} (-D)^{k-\frac{\mathbf{rk}(\underline{L})}{2}-1} \times \sum_{c\geq 1} c^{-k}\left(H_{\underline{L},c}(r,D,t) + (-1)^k H_{\underline{L},c}(-r,D,t)\right),$$

where $H_{\underline{L},c}(r,D,t)$ is the lattice sum

$$\sum_{\lambda\in L/cL, d\in(\mathbb{Z}/c\mathbb{Z})^\times} e_c\left(\beta(\lambda+r)d^{-1} + (\beta(t)-D)d + \beta(t,\lambda+r)\right).$$

- they were defined for Jacobi forms of scalar index by Eichler & Zagier (1985), where they were used to develop a theory of newforms

Definition

For every l in $\mathbb{N}$, define the operator $U(l)$ on the space $J_{k,\underline{L}}$ as:

$$\phi|U(l)(\tau,z) := \phi(\tau, lz).$$

Definition

For every l in $\mathbb{N}$, define the operator $V(l)$ on the space $J_{k,\underline{L}}$ as:

$$\phi|V(l)(\tau,z) = l^{\frac{k}{2}-1} \sum_{\substack{M \in \Gamma \backslash \mathcal{M}_2(\mathbb{Z}) \\ \det(M)=l}} (\phi|_{k,\underline{L}} M)\,|U(\sqrt{l})(\tau,z).$$

- these operators were defined by Gritsenko (1988) as double coset operators and $V(\cdot)$ was also defined and used in Cléry & Gritsenko (2013)

- the operator $U(l)$ corresponds to the endomorphism "multiplication by l" on $\mathcal{T}_{\tau,\underline{L}} = (L \otimes \mathbb{C})/(L\tau \oplus L)$
- assume that $L\tau \oplus L$ is contained in L' with index l; if $\{\omega_1, \omega_2\}$ is a basis for L', then there exists $M \in \mathcal{M}_2(\mathbb{Z})$ with determinant l, such that $\left(\begin{smallmatrix} \tau \\ 1 \end{smallmatrix}\right) = M \left(\begin{smallmatrix} \omega_1 \\ \omega_2 \end{smallmatrix}\right)$
- if $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then $(\phi|_{k,\underline{L}} M) \,|\, U(\sqrt{l})(\tau, z)$ contains a factor of $\phi\left(M\tau, \frac{lz}{c\tau+d}\right)$
- think of $U(l) : M_k(N) \to M_k(lN)$,

$$U(l)f(\tau) = \sum a(ln)q^n$$

and of $V(l) : M_k(N) \to M_k(lN)$

$$V(l)f(\tau) = \sum a(n)q^{ln}$$

- since Jacobi–Eisenstein series are perpendicular to cusp forms, we obtain the following decomposition:

$$J_{k,\underline{L}} = S_{k,\underline{L}} \oplus J_{k,\underline{L}}^{Eis}$$

- we are interested in a theory of *newforms* with respect to Hecke operators (defined by Ajouz (2015)): for every l in $\mathbb{N}$, which is coprime to $\mathrm{lev}(\underline{L})$, define a double coset operator on $J_{k,\underline{L}}$

$$T_0(l)\phi(\tau,z) := l^{k-2-rk(\underline{L})} \sum_{g \in J_{\underline{L}}(\mathbb{Z}) \backslash J_{\underline{L}}(\mathbb{Z}) \left(\begin{smallmatrix} l^{-1} & 0 \\ 0 & l \end{smallmatrix}\right) J_{\underline{L}}(\mathbb{Z})} \phi|_{k,\underline{L}} g(\tau,z)$$

- the *twists* of $E_{k,\underline{L},r}$ by primitive Dirichlet characters modulo divisors of N_r (order of r in $L^\#/L$) form a *basis of eigenforms* of $J_{k,\underline{L}}^{Eis}$

$$E_{k,\underline{L},r,\chi}(\tau,z) := \sum_{d \in \mathbb{Z}_{N_r}^\times} \chi(d) E_{k,\underline{L},dr}(\tau,z)$$

- there are two ways to think about newforms:
  - Jacobi forms coming from lattices of *lower level*

    e.g. $J_{k,\underline{L}} \simeq \mathfrak{M}_{2k-1-\mathrm{rk}(\underline{L})}(\mathrm{lev}(\underline{L})/4)^{-}$ for $\underline{L} \simeq (\mathbb{Z}, (x,y) \mapsto \det(\underline{L})xy)$
  - Jacobi forms coming from *sublattices* (i.e. $\underline{L} = \underline{L}_1 \oplus \underline{L}_2$)
- linear operators give *structure* to a finite dimensional vector space (think basis of Hecke eigenforms) and $U(\cdot)$ and $V(\cdot)$ are "predecesors" of $T(\cdot)$
- they facilitate *lifts* between different types of modular forms; for example, $V(\cdot)$ was used by Cléry & Gritsenko (2013) to construct maps

$$J_{k,\underline{L}(m)}(\xi) \to M_k\left(\tilde{O}^{+}(L')\right),$$

  where $\underline{L}(m) = (L, m \cdot \beta)$
- they have algebraic interpretations in terms of the surfaces that our modular forms underlie
- the level raising operators satisfy the following properties:

Proposition (M., 2016)

The operators $U(l)$ map $J_{k,\underline{L}}$ to $J_{k,\underline{L}(l^2)}$ and the $V(l)$ map $J_{k,\underline{L}}$ to $J_{k,\underline{L}(l)}$. Moreover, if ϕ in $J_{k,\underline{L}}$ has the Fourier expansion

$$\phi(\tau, z) = \sum_{(D,r)\in \mathrm{supp}(\underline{L})} C(D, r) e((\beta(r) - D)\tau + \beta(r, z)),$$

then $\phi|U(l)$ and $\phi|V(l)$ have the following Fourier expansions:

$$\phi|U(l)(\tau, z) = \sum_{\substack{(D,r')\in \mathrm{supp}(\underline{L}(l^2)) \\ r' \in lL(l^2)^\#}} C(D, lr') e\left((l^2\beta(r') - D)\tau + l^2\beta(r', z)\right)$$

$$\phi|V(l)(\tau, z) = \sum_{(D,r'')\in \mathrm{supp}(\underline{L}(l))} \sum_{\substack{a|(l\beta(r'')-D),l \\ \frac{r''}{a} \in L(l)^\#}} a^{k-1} C\left(\frac{lD}{a^2}, \frac{lr''}{a}\right),$$

$$\times\, e\left((l\beta(r'') - D)\tau + l\beta(r'', z)\right).$$

- note that the level of $\underline{L}(m)$ is $\mathrm{lev}(\underline{L}(m)) = m \cdot \mathrm{lev}(\underline{L})$.

- The proof is straight forward:
  - modularity of $\phi|U(l)$ follow easily from that of ϕ; for $\phi|V(l)$, a clever choice of coset representatives does the job
  - to determine the Fourier expansions, use the fact that $L^\# \simeq L(m)^\#$ as $\mathbb{Z}$-modules ($r \mapsto \frac{1}{m}r$)

Proposition (M., 2017)

The operators $U(\cdot)$ and $V(\cdot)$ commute:

$$\phi|U(l)|U(l') = \phi|U(ll') \tag{1}$$

$$\phi|V(l')|U(l) = \phi|U(l)|V(l') \tag{2}$$

$$\phi|V(l')|V(l) = \sum_{d|\gcd(l,l')} d^{k-1}\phi|V\left(\frac{ll'}{d^2}\right)|U(d) \tag{3}$$

The operators $U(\cdot)$ and $V(\cdot)$ commute with $T(\cdot)$. If l is coprime to lev($\underline{L}$) and to l', then:

$$T(l)[\phi|U(l')] = [T(l)\phi]|U(l') \tag{4}$$

$$T(l)[\phi|V(l')] = [T(l)\phi]|V(l'). \tag{5}$$

- the proof of (1) follows from the definition; reminder: $\phi|U(l)(\tau,z) = \phi(\tau, lz)$
- (2) follows from (1); reminder: $V(l)$ depends on $U(\sqrt{l})$
- $(3), (4)$ and (5) can be proved by comparing Fourier coefficients of both sides:
  - (3) is, as stated in *The Theory in Jacobi forms*, a question of counting: compute $N(e)$ from

$$\sum_{\substack{b|\gcd(n,l)\\ \frac{r}{b}\in L^{\#}}} b^{k-1} \sum_{\substack{a|\gcd\left(\frac{nl}{b^2},l'\right)\\ \frac{r}{ab}\in L^{\#}}} a^{k-1} c\left(\frac{nll'}{a^2b^2}, \frac{ll'r}{ab}\right) = \sum_e N(e) e^{k-1} c\left(\frac{nll'}{e^2}, \frac{ll'r}{e}\right)$$

$$\implies N(e) = \#\left\{d : d \mid \left(n, l, l', e, \frac{nl}{e}, \frac{nl'}{e}, \frac{ll'}{e}, \frac{nll'}{e^2}\right) \text{ and } \frac{r}{e} \in L^{\#}\right\}$$

  - for (4) and (5), use brute force combined with some modular arithmetic, which uses the fact that $(l,l') = 1$ and that $\mathrm{lev}(\underline{L})L^{\#} \subseteq L$ for all L

- the goal would be to show that $J^{Eis}_{k,\underline{L}}$ is the image of Eisenstein series coming from lower level under combinations of $U(\cdot)$ and $V(\cdot)$ + some *new* Eisenstein series and use it in

$$J_{k,\underline{L}} = S_{k,\underline{L}} \bigoplus J^{Eis}_{k,\underline{L}}$$

Theorem (M., 2018)

Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice. For every r in Iso($\underline{L}$) and every l in $\mathbb{N}$, the following holds:

$$E_{k,\underline{L},r}|U(l)(\tau, z) = \sum_{\substack{s\in\left(\frac{1}{l}L^{\#}\right)/L \\ ls\equiv r \bmod L}} E_{k,\underline{L}(l^2),s}(\tau, z),$$

$$E_{k,\underline{L},r}|V(l)(\tau, z) = \sum_{s\in\mathrm{Iso}(\underline{L}(l))} E_{k,\underline{L}(l),s}(\tau, z) \sum_{\substack{a|(l\beta(s),l) \\ \frac{ls}{a}\equiv r \bmod L}} a^{k-1}.$$

- we saw that the *singular term* (terms with $D = 0$ in the Fourier expansion) of $E_{k,\underline{L},r}$ is given by

$$\frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z) + (-1)^k \vartheta_{\underline{L},-r}(\tau,z)\right)$$

- the theta functions $\{\vartheta_{\underline{L},r} : r \in \mathrm{Iso}(\underline{L})\}$ are linearly independent for a fixed $\underline{L}$

$$\vartheta_{\underline{L},r}(\tau,z) := \sum_{\substack{s \in L^{\#} \\ s \equiv r \bmod L}} e\left(\beta(s)\tau + \beta(s,z)\right)$$

- for every $\underline{L}$ and every ϕ in $J_{k,\underline{L}}$, we have $C_\phi(D,r) = C_\phi(D',r')$ whenever $D = D'$ and $r \equiv r' \bmod L$, due to the invariance of ϕ with respect to the $|_{k,\underline{L}}$ action of L^2

- in particular, this implies that

$$U(l) : J^{\mathrm{Eis}}_{k,\underline{L}} \to J^{\mathrm{Eis}}_{k,\underline{L}(l^2)} \text{ and}$$
$$V(l) : J^{\mathrm{Eis}}_{k,\underline{L}} \to J^{\mathrm{Eis}}_{k,\underline{L}(l)}$$

- this result can also be used to show the following:

Proposition (M., 2018)

For every r in $Iso(\underline{L})$, every l in $\mathbb{N}$ and every d in $\mathbb{N}$ such that $d^2 \mid l$, the following holds:

$$E_{k,\underline{L},r}|U(d)|V\left(\frac{l}{d^2}\right)(\tau,z) = \sum_{x \in \mathrm{Iso}(\underline{L}(l))} E_{k,\underline{L}(l),x}(\tau,z) \sum_{\substack{a \mid \left(l\beta(x), \frac{l}{d^2}\right) \\ \frac{lx}{da} \equiv r \bmod L}} a^{k-1}.$$

- to prove this, apply the previous theorem and then show that

$$\sum_{\substack{s \in \left(\frac{1}{d}L^{\#}\right)/L \\ ds \equiv r \bmod L}} \delta\left(s, \frac{lx}{d^2 a}\right) = \delta\left(r, \frac{lx}{da}\right)$$

Corollary

For every l in $\mathbb{N}$, the following holds:

$$\frac{1}{l^{k-1}}\prod_{p|l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2|l}\mu(d)E_{k,\underline{L},0}|U(d)|V\left(\frac{l}{d^2}\right)$$
$$=\frac{1}{l^{k-1}}\prod_{p|l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2|l}\mu(d)\sum_{\substack{x\in\left(\frac{1}{l}L\right)/L\\ l\beta(x)\in\mathbb{Z}}}E_{k,\underline{L}(l),x}\sum_{\substack{a|\left(l\beta(x),\frac{l}{d^2}\right)\\ \frac{lx}{da}\in L}}a^{k-1}.$$

- this is nothing but the previous Proposition with $r=0$, the observation that

$$\{x\in\underline{L}(l)^{\#}/L : lx\in L\}=\left(\frac{1}{l}L\right)/L$$

and a strange looking "normalizing" factor; it is normalizing, because
  - when k is odd, both sides of the above equation vanish (this is because $E_{k,\underline{L},r}=(-1)^k E_{k,\underline{L},-r}$)
  - when k is even, the coefficient corresponding to $E_{k,\underline{L}(l),0}$ on the right-hand side of the above equation is equal to one, in other words

$$\frac{1}{l^{k-1}}\prod_{p|l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2|l}\mu(d)\sigma_{k-1}\left(\frac{l}{d^2}\right)=1$$

- show that every $E_{k,\underline{L},r}$ can be written as linear combinations of $E_{k,\underline{L}(1/ll'^2),x}|U(l')|V(l)$
- which are the "new" Eisenstein series?

Thank you!

- show that every $E_{k,\underline{L},r}$ can be written as linear combinations of $E_{k,\underline{L}(1/ll'^2),x}|U(l')|V(l)$
- which are the "new" Eisenstein series?

Questions?