On the connection between Jacobi forms and elliptic modular forms

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Structure of the talk

1. Setup
2. Jacobi forms and elliptic modular forms
3. Jacobi forms of index D_n

1. Setup

- $\mathrm{SL}_2(\mathbb{Z}) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{M}_2(\mathbb{Z}) : ad - bc = 1 \right\}$
- $m \in \mathbb{N}$, $\Gamma_0(m) := \mathrm{SL}_2(\mathbb{Z}) \cap \left\{ \left(\begin{smallmatrix} * & * \\ 0 \bmod m & * \end{smallmatrix}\right) \right\}$
- $\mathfrak{H} := \{ z \in \mathbb{C} : \Im(z) > 0 \}$
- $\mathrm{SL}_2(\mathbb{Z}) \circlearrowright \mathfrak{H}$: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau := \frac{a\tau + b}{c\tau + d}$

Definition

Fix k in $\mathbb{N}$. An *elliptic modular form of weight k with respect to $\Gamma_0(m)$* is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ which satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

- $M_k(m) := \mathrm{Span}_{\mathbb{C}}\{f \text{ as above}\}$ is *finite-dimensional*
- $e(x) := e^{2\pi i x}$, every f as above has a Fourier expansion:

$$f(\tau) = \sum_{n \geqslant 0} a_f(n)\, e(n\tau)$$

- $S_k(m) := \{ f \in M_k(m) : a_f(0) = 0 \}$

- fix $\underline{L} = (L, \beta)$:
  - $L \simeq \mathbb{Z}^{\mathrm{rk}(\underline{L})}$
  - $\beta : L \times L \to \mathbb{Z}$ is
    - *positive-definite*: $\beta(\lambda, \lambda) > 0$ for all λ in L
    - *even*: $\beta(\lambda) := \frac{\beta(\lambda,\lambda)}{2} \in \mathbb{Z}$
- $J^{\underline{L}} := \mathrm{SL}_2(\mathbb{Z}) \ltimes L^2$
- $(\tau, \mathfrak{z}) \in \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$
- $J^{\underline{L}} \circlearrowright \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$: $\left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)\right](\tau, \mathfrak{z}) := \left(\frac{a\tau+b}{c\tau+d}, \frac{\mathfrak{z}+\tau\lambda+\mu}{c\tau+d}\right)$
- $L^{\#} := \{t \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(t, \lambda) \in \mathbb{Z}$ for all λ in $L\}$

Definition

Fix k in $\mathbb{N}$. A *Jacobi form of weight k and index $\underline{L}$* is a holomorphic function $\varphi : \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ which satisfies

1. $$\varphi(\gamma(\tau, \mathfrak{z})) = (c\tau + d)^k \, e\left(\frac{c\beta(\mathfrak{z}+\lambda\tau+\mu)}{c\tau+d} - \tau\beta(\lambda) - \beta(\lambda, \mathfrak{z})\right) \times \varphi(\tau, \mathfrak{z})$$
2. $$\varphi(\tau, \mathfrak{z}) = \sum_{\substack{D \in \mathbb{Q}_{\leqslant 0}, t \in L^{\#} \\ D \equiv \beta(t) \bmod \mathbb{Z}}} C_\varphi(D, t) \, e((\beta(t) - D)\tau + \beta(t, \mathfrak{z})).$$

- $J_{k,\underline{L}} := \mathrm{Span}_{\mathbb{C}}\{\varphi \text{ as above}\}$ is *finite-dimensional*
- $S_{k,\underline{L}} := \left\{\varphi \in J_{k,\underline{L}} : C_\varphi(0,t) = 0, \forall t \in L^\# \text{ s.t. } \beta(t) \in \mathbb{Z}\right\}$

Example

If $\underline{L}$ is *unimodular*, then

$$\vartheta_{\underline{L}}(\tau, \mathfrak{z}) := \sum_{\lambda \in L} e\left(\beta(\lambda)\tau + \beta(\lambda, \mathfrak{z})\right)$$

is a Jacobi form of *weight* $\frac{\mathrm{rk}(\underline{L})}{2}$ and *index* $\underline{L}$.

- set $\Gamma_\infty := \{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\} \leqslant \Gamma_0(m)$ and define the *weight k Eisenstein series*

$$E_k(\tau) := \frac{1}{2} \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma} (c\tau + d)^{-k} \in M_k(m)$$

- $M_k(1) = S_k(1) \oplus \mathbb{C}E_k$

- set $J_\infty^{\underline{L}} := \{[\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right), (0,\mu)] : n \in \mathbb{Z}, \mu \in L\} \leqslant J^{\underline{L}}$
- define $g_{\underline{L},t}(\tau, \mathfrak{z}) := e(\beta(t)\tau + \beta(t,z))$ $(t \in L^{\#})$
- for every r in $L^{\#}$ s.t. $\beta(r) \in \mathbb{Z}$, define the *Eisenstein series of weight k and index $\underline{L}$ associated with r*

$$E_{k,\underline{L},r}(\tau,\mathfrak{z}) := \frac{1}{2} \sum_{\gamma \in J_\infty^{\underline{L}} \backslash J^{\underline{L}}} g_{\underline{L},r}(\gamma(\tau,\mathfrak{z}))(c\tau+d)^{-k} \times e\left(\frac{-c\beta(\mathfrak{z}+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,\mathfrak{z})\right)$$

- $J_{k,\underline{L}}^{\mathrm{Eis}} := \mathrm{Span}_{\mathbb{C}}\left\{E_{k,\underline{L},r} : r \in L^{\#}/L \text{ and } \beta(r) \in \mathbb{Z}\right\}$

Theorem (M. 2017)

When $k > \frac{rk(\underline{L})}{2} + 2$, the Eisenstein series are elements of $J_{k,\underline{L}}$ and

$$J_{k,\underline{L}} = S_{k,\underline{L}} \oplus J_{k,\underline{L}}^{\mathrm{Eis}}.$$

Hecke theory

- (Atkin–Lehner, 1970): $T(l) : M_k(m) \to M_k(m)$ $(l \in \mathbb{N})$
  - $T(m)T(n)f = T(n)T(m)f$
  - $\langle T(l)f, g\rangle = \langle f, T(l)g\rangle$
  - $T(l) : S_k(m) \to S_k(m)$
- (Ajouz, 2015): $T(l) : J_{k,\underline{L}} \to J_{k,\underline{L}}$
  - $T(m)T(n)\varphi = T(n)T(m)\varphi$
  - $\langle T(l)\varphi, \psi\rangle = \langle \varphi, T(l)\psi\rangle$
  - $T(l) : S_{k,\underline{L}} \to S_{k,\underline{L}}$
- a *Hecke eigenform* is f in $M_k(m)$ s.t. $T(l)f = \lambda_f(l)f$
- a *Hecke eigenform* is φ in $J_{k,\underline{L}}$ s.t. $T(l)\varphi = \lambda_\varphi(l)\varphi$

Theorem

The space $M_k(m)$ has a basis of Hecke eigenforms.

Theorem

The space $J_{k,\underline{L}}$ has a basis of Hecke eigenforms.

- $n \mid m$ and $d \mid \frac{m}{n}$, $g \in M_k(n) \implies g(d\tau) \in M_k(m)$
- $M_k(m) = M_k^{\text{new}}(m) \oplus M_k^{\text{old}}(m)$, compatible with Hecke ops.
- a *normalized newform* is a Hecke eigenform in $S_k^{\text{new}}(m)$ s.t. $a_f(1) = 1$
  - if f is a normalized newform, then $f(\tau) = \sum_{n \geqslant 1} \lambda_f(n) e(n\tau)$

Remark

There does not exist a complete theory of newforms for Jacobi forms of lattice index.

Modularity Theorem (Wiles *et. al.*)

Elliptic curves *over* $\mathbb{Q}$ *are "related to" normalized newforms of weight* 2.

Fermat's Last Theorem (Wiles, 1994)

No three positive integers a, b, *and* c *satisfy the equation* $a^n + b^n = c^n$ *for any integer* $n \geqslant 3$.

2. Jacobi forms and elliptic modular forms

- if f is a Hecke eigenform, then $L(s,f) := \sum_{n\geqslant 1} \lambda_f(n) n^{-s}$
- if φ is a Hecke eigenform, then $L(s,\varphi) := \sum_{n\geqslant 1} \lambda_\varphi(n) n^{-s}$

Langlands Program

It is a series of conjectures about connections between geometry and number theory.

$$\textit{(arithmetic } L\textit{-functions)} \leftrightsquigarrow \textit{(automorphic } L\textit{-functions)}$$

- $f \in M_t(m)$ is a Hecke eigenform:

$$L(s,f) = \prod_{p \text{ prime}} (1 - \lambda_f(p)p^{-s} + p^{t-1-2s})^{-1}$$

- $\mathrm{rk}(\underline{L})$ is odd and $\varphi \in J_{k,\underline{L}}$ is a Hecke eigenform:

$$L(s,\varphi) = \prod_{p \text{ prime}} (1 - \lambda_\varphi(p)p^{-s} + p^{2k-\mathrm{rk}(\underline{L})-2-2s})^{-1}$$

Liftings

- $\mathrm{lev}(\underline{L}) := \min\{N \in \mathbb{N} : N\beta(t) \in \mathbb{Z}, \forall t \text{ in } L^{\#}\}$

Theorem (Ajouz, 2015)

If rk($\underline{L}$) is odd and $2k \geqslant$ rk($\underline{L}$) $+ 3$, then there exist liftings from $S_{k,\underline{L}}$ to $M_{2k-\mathrm{rk}(\underline{L})-1}(\mathrm{lev}(\underline{L})/2)$ which commute with Hecke operators.

- $M_k(m) = M_k^-(m) \oplus M_k^+(m)$

$$f \in M_k^{\varepsilon}(m) \implies L_m(s,f) = \varepsilon L_m(k-s,f)$$

Birch–Swinnerton-Dyer Conjecture

If E is an elliptic curve, then the rank of the group of rational points of E is the order of the zero of $L(s,E)$ at $s = 1$.

- $\mathfrak{M}_k(m) := M_k^{new}(m) \oplus \{\text{very special oldforms}\}$
- $\mathfrak{M}_k^{\varepsilon}(m) := \mathfrak{M}_k(m) \cap M_k^{\varepsilon}(m)$

Conjecture ($\star$) (Ajouz, 2015)

If rk($\underline{L}$) is odd, then there exists an isomorphism

$$J_{k,\underline{L}} \simeq \mathfrak{M}^{\varepsilon}_{2k-rk(\underline{L})-1}(\mathrm{lev}(\underline{L})/4)$$

which commutes with Hecke operators, where $\varepsilon = -$ if rk($\underline{L}$) $\equiv 1$ or $3 \bmod 8$ and $\varepsilon = +$ otherwise.

- ($\star$) was proved for rk($\underline{L}$) $= 1$ by Skoruppa–Zagier (1988)
- $$\begin{tikzcd} M_{2k-2}(1) \arrow[d, "\text{Shimura}"', "\wr"] & M_k^*(\Gamma_2) \arrow[l, "\sim", "\text{S–K}"'] \\ M^+_{k-\frac{1}{2}}(4) \arrow[r, "\sim", "\text{Eichler–Zagier}"'] & J_{k,1} \arrow[u, "\wr", "\text{Maaß}"'] \end{tikzcd}$$
- can *lift* Jacobi forms to *reflective modular forms* (RMFs)
  - some RMFs are *automorphic discriminants* of moduli spaces
  - Fourier coefficients of a RMF $\rightsquigarrow$ generators and relations of Lorentzian Kac–Moody algebras

3. Jacobi forms of index D_n

- $D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \in 2\mathbb{Z}\}$
- *Euclidean bilinear form*: $(x, y) := x_1 y_1 + \cdots + x_n y_n$
- n is odd

Fact

If $m \equiv n \bmod 8$, *then* $J_{k+[\frac{n}{2}],D_n} \simeq J_{k+[\frac{m}{2}],D_m}$.

- enough to consider $n = 1, 3, 5$ and 7

Fact

There exists a differential operator $\partial : J_{k,\underline{L}} \to J_{k+2,\underline{L}}$.

Fact

If $f \in M_{k_1}(1)$ *and* $\varphi \in J_{k_2,\underline{L}}$, *then* $f\varphi \in J_{k_1+k_2,\underline{L}}$.

Generators for even weights

- $E_8 = \left\{ x : x \in \mathbb{Z}^8 \text{ or } x \in \left(\mathbb{Z} + \frac{1}{2}\right)^8, x_1 + \cdots + x_8 \in 2\mathbb{Z} \right\}$
- $\alpha_n : D_n \to E_8,\ (x_1, \ldots, x_n) \mapsto (0, \ldots, 0, x_1, \ldots, x_n)$

Fact

If $\varphi \in J_{k,E_8}$, then $\varphi(\tau, \alpha_n(\mathfrak{z})) \in J_{k,D_n}$.

- consider $\vartheta_{E_8}(\tau, \mathfrak{z}) = \sum_{\lambda \in E_8} e\left(\frac{(\lambda,\lambda)}{2}\tau + (\lambda, \mathfrak{z})\right)$ and define

$$E_{4,D_n}(\tau, \mathfrak{z}) := \vartheta_{E_8}(\tau, \alpha_n(\mathfrak{z}))$$
$$E_{6,D_n}(\tau, \mathfrak{z}) := \partial\vartheta_{E_8}(\tau, \alpha_n(\mathfrak{z}))$$
$$E_{8,D_n}(\tau, \mathfrak{z}) := \partial^2\vartheta_{E_8}(\tau, \alpha_n(\mathfrak{z}))$$

Theorem (Boylan–Skoruppa, in preparation)

$$J_{2k,D_n} = M_{2k-4}(1)E_{4,D_n} \oplus M_{2k-6}(1)E_{6,D_n} \oplus M_{2k-8}(1)E_{8,D_n}$$

Generators for odd weights

- $\eta(\tau) := e(\tau/24) \prod_{n \geqslant 1} (1 - e(n\tau))$ has weight $\frac{1}{2}$
- $\vartheta(\tau, z) := \sum_{n \in \mathbb{Z}} \left(\frac{-4}{n}\right) e\left(\tau \frac{n^2}{8} + \frac{nz}{2}\right)$ has weight $\frac{1}{2}$ and index $\left(\mathbb{Z}, (x, y) \mapsto 2 \times \frac{1}{2} xy\right)$

Fact

For $1 \leqslant n \leqslant 7$, the function

$$\psi_{12-n,D_n}(\tau, \mathfrak{z}) := \eta(\tau)^{24-3n} \vartheta(\tau, z_1) \ldots \vartheta(\tau, z_n)$$

is an element of S_{12-n,D_n} ($\mathfrak{z} = (z_1, \ldots, z_n)$).

Theorem (Boylan–Skoruppa, in preparation)

$$J_{2k+1,D_n} = M_{2k-11+n}(1) \psi_{12-n,D_n}$$

Strategy

- $\mathrm{lev}(D_n) = 8$

$$(\star) \implies J_{k,D_n} \simeq \begin{cases} \mathfrak{M}^-_{2k-n-1}(2), & n = 1,3 \\ \mathfrak{M}^+_{2k-n-1}(2), & n = 5,7 \end{cases}$$

- for fixed odd n and every weight:
  1. find formulas for *Fourier coefficients* of generators (by hand)
  2. find basis of *Hecke eigenforms* (by hand/use SageMath)
  3. if φ is a Hecke eigenform: $\lambda_\varphi(l) = \dfrac{T(l)\varphi}{\varphi} = \dfrac{C_{T(l)\varphi}(D,r)}{C_\varphi(D,r)}$

     (implement item 1 in SageMath)
  4. *compare* eigenvalues with those of elliptic modular forms (available on the LMFDB)

Remark

Conjecture $(\star) \implies J_{k,D_1} \simeq J_{k+1,D_3}$ and $J_{k,D_5} \simeq J_{k+1,D_7}$.
But Boylan–Skoruppa $\implies$ this is false!

- e.g. J_{11,D_1} is 1-dim and J_{12,D_3} is 3-dim

Weight 4

- $J_{4,D_n} \simeq \mathbb{C}E_{4,D_n}$
  1. Fourier coefficients are *representation numbers* of quadratic forms and
  $$E_{4,D_n} = E_{4,D_n,0}$$
  2. ✓
  3. $(D,r) = (-1,(0,0,0))$ in $\lambda_{E_{4,D_3}}(l) = \frac{C_{T(l)E_{4,D_3}}(D,r)}{C_{E_{4,D_3}}(D,r)}$ gives

l	1	3	5	7	9	11	13	15
$\lambda_{E_{4,D_3}}(l)$	1	28	126	344	757	1332	2198	3528

  4. the eigenvalues match

$$\begin{aligned} E_4(\tau) =& 1/240 + e(\tau) + 9\,e(2\tau) + 28\,e(3\tau) + 73\,e(4\tau) + 126\,e(5\tau) \\ &+ 252\,e(6\tau) + 344\,e(7\tau) + 585\,e(8\tau) + 757\,e(9\tau) \\ &+ 1134\,e(10\tau) + 1332\,e(11\tau) + 2044\,e(12\tau) + 2198\,e(13\tau) \\ &+ 3096\,e(14\tau) + 3524\,e(15\tau) + \dots \end{aligned}$$

Summary

- $J^{old}_{k,D_n} := \{f \in J_{k,D_n}$: eigenv. of f match those of ell. oldforms$\}$
- $J^{new}_{k,D_n} := \{f \in J_{k,D_n}$: eigenv. of f match those of ell. newforms$\}$
- results in The Table suggest that

$$\begin{aligned}
J^{old}_{k+1,D_1} &\simeq J^{old}_{k+3,D_5} \simeq M^-_{2k}(1) = \mathfrak{M}^{old,-}_{2k}(2)\\
J^{old}_{k+2,D_3} &\simeq J^{old}_{k+4,D_7} \simeq M^+_{2k}(1) = \mathfrak{M}^{old,+}_{2k}(2)\\
J^{new}_{k+1,D_1} &\simeq J^{new}_{k+2,D_3} \simeq M^{new,-}_{2k}(2) = \mathfrak{M}^{new,-}_{2k}(2),\\
J^{new}_{k+3,D_5} &\simeq J^{new}_{k+4,D_7} \simeq M^{new,+}_{2k}(2) = \mathfrak{M}^{new,+}_{2k}(2)
\end{aligned}$$

- Skoruppa–Zagier (1988):

$$J_{k+1,D_1} \simeq \mathfrak{M}^-_{2k}(2) = M^{new,-}_{2k}(2) \oplus M^-_{2k}(1)$$

Conjecture (M. 2019)

For every $k \geqslant 2$, the following holds:

$$\begin{aligned} J_{k+2,D_3} &\simeq \mathfrak{M}_{2k}^{new,-}(2) \oplus \mathfrak{M}_{2k}^{old,+}(2), \\ J_{k+3,D_5} &\simeq \mathfrak{M}_{2k}^{new,+}(2) \oplus \mathfrak{M}_{2k}^{old,-}(2), \\ J_{k+4,D_7} &\simeq \mathfrak{M}_{2k}^{+}(2) \end{aligned}$$

and these isomorphisms are Hecke equivariant.

Lemma (M. 2019)

For every $k \geqslant 2$, the following holds:

$$\begin{aligned} \dim(J_{k+2,D_3}) &= \dim(\mathfrak{M}_{2k}^{new,-}(2)) + \dim(\mathfrak{M}_{2k}^{old,+}(2)), \\ \dim(J_{k+3,D_5}) &= \dim(\mathfrak{M}_{2k}^{new,+}(2)) + \dim(\mathfrak{M}_{2k}^{old,-}(2)) \text{ and} \\ \dim(J_{k+4,D_7}) &= \dim(\mathfrak{M}_{2k}^{+}(2)). \end{aligned}$$

The proof of the lemma uses:

- dimension formulas for $M_{2k}(1)$, from which in turn we obtain dimension formulas for J_{t,D_n}
- dimension formulas for $M_{2k}(2)$, from which in turn we obtain dimension formulas for $M_{2k}(2)^{new} = S_{2k}(2)^{new}$
- the fact that $M_{2k}(2)^{new} = M_{2k}(2)^{new,+} \oplus M_{2k}(2)^{new,-}$, combined with the formula

$$\dim(M_{2k}^{new,+}(2)) = \begin{cases} \dim(M_{2k}^{new,-}(2)), & k \equiv 2, 3 \bmod 4, \\ \dim(M_{2k}^{new,-}(2)) + 1, & \text{otherwise} \end{cases}$$

To do

- make experimental results precise
- more computations
- newform theory
- trace formula for Jacobi forms of lattice index

Thank you!