UNIVERSITY OF NOTTINGHAM

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On Jacobi forms of lattice index

Andreea Mocanu

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Abstract

Jacobi forms arise naturally in number theory in several ways: theta series arise as functions of lattices, Siegel modular forms give rise to Jacobi forms through their Fourier–Jacobi expansion and the largest Mathieu group gives rise to semi-holomorphic Maaß–Jacobi forms, for example. Jacobi forms of lattice index have applications in the theory of reflective modular forms and that of vertex operator algebras, among other areas.

Poincaré and Eisenstein series are building blocks for every type of automorphic forms. We define Poincaré series for Jacobi forms of lattice index and show that they reproduce Fourier coefficients of cusp forms under the Petersson scalar product. We compute the Fourier expansions of Poincaré and Eisenstein series and give an explicit formula for the Fourier coefficients of the trivial Eisenstein series in terms of values of Dirichlet L-functions at negative integers. For even weight and fixed index, we obtain non-trivial linear relations between the Fourier coefficients of non-trivial Eisenstein series and those of the trivial one. This result is used to obtain formulas for the Fourier coefficients of Eisenstein series associated with isotropic elements of small order.

A more efficient way of breaking down a given space of automorphic forms is into its oldspace and its newspace. We study the linear operators leading to a theory of newforms for Jacobi forms of lattice index, namely Hecke operators, operators arising from the action of the orthogonal group of the discriminant module associated with the lattice in the index and level raising operators. We show that these operators commute with one another and are therefore suitable to define a newform theory. We define the level raising operators of type $U(I)$ (for every isotropic subgroup I of the discriminant module associated with the lattice in the index) and show that they preserve cusp forms and Eisenstein series. We give a formula for the action of the level raising operators $U(I)$ and $V(l)$ and operators $W(s)$ arising from the action of the orthogonal group on cusp forms and Eisenstein series. We obtain a description of some of the oldforms in a given space of Jacobi forms using these operators and the relation between Jacobi forms and vector-valued modular forms for the dual of the Weil representation.

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Contents

Introduction

This thesis concerns a certain generalization of elliptic modular forms called Jacobi forms of lattice index. Interest in Jacobi forms has increased in recent years due to their numerous applications to number theory, algebraic geometry and string theory. Computing Jacobi forms gives direct information on the Fourier coefficients of half-integral weight modular forms [**RSST16**], they play a part in the Mirror Symmetry conjecture for K3 surfaces [**GN96**] and a certain type of Jacobi forms can be the elliptic genus of Calabi–Yau manifolds [**Gri91**], to name some of these applications.

The arithmetic theory of Jacobi forms of *scalar index* was established in [**EZ85**]. In this book, the authors analyse Eisenstein series and cusp forms, compute the Taylor expansions of Jacobi forms, define Hecke operators on these functions and, last but not least, discuss the relation between Jacobi forms and half-integral weight elliptic modular forms, vector-valued modular forms and Siegel modular forms. Since then, several generalizations of Jacobi forms have been studied, such as Siegel–Jacobi forms [**Zie89**], Jacobi forms of lattice index [**Gri88**] or Jacobi forms over number fields [**Boy15**].

Let k be a positive integer and let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$. By a *Jacobi form of weight k and index $\underline{L}$* we mean a holomorhic function of two variables (a *modular* variable denoted by τ and an *abelian* variable denoted by z), which is invariant with respect to a certain action of the integral Jacobi group associated with $\underline{L}$ and which has a prescribed Fourier expansion (see Definition 1.23). These functions were introduced in [**Gri88**] as the Fourier–Jacobi coefficients of orthogonal modular forms. They have applications in the theory of reflective modular forms [**Gri18**] and that of vertex operator algebras [**KM15**], among other areas.

Our goal is to investigate the relation between Jacobi forms of lattice index and elliptic modular forms. This would enable the transfer of information and mathematical tools from one side to the other. Lifts of Jacobi forms to other type of automorphic forms often have special properties, for example their Fourier coefficients satisfy simple linear relations [**Maa79**], or their L-functions satisfy certain vanishing properties [**EZ85**].

This problem was solved for Jacobi forms of scalar index in [**SZ88**], where algebraic lifting maps were defined between the former and elliptic modular forms, using the action of Hecke operators on the Fourier coefficients of Jacobi forms. Hecke operators were defined in [**EZ85**] only for good primes (i.e. primes not dividing the index). As a result, the proof that a linear combination of these maps defines an isomorphism between Jacobi forms of scalar index and *a certain space* of elliptic modular forms utilizes heavy tools, such as trace formulas and a theory of newforms on either side. To this end, in this thesis we study the linear operators which should lead to a theory of newforms for Jacobi forms of lattice index, namely Hecke operators, operators arising from the action of the orthogonal group of the discriminant module associated with the lattice in the index (see Chapter 3) and level raising operators (see Chapter 4).

In [**Bri06**], the author constructs lifting maps similar to those in [**SZ88**] from spaces of Jacobi forms of *matrix index* to spaces of elliptic modular forms. In addition, they define maps in the opposite direction and prove that, when the dimension of the matrix in the index is congruent to 1 modulo 8, these maps are adjoint with respect to the

Petersson scalar products on the two underlying spaces. The proof relies on the construction of a holomorphic kernel function for the two maps and on the fact that this kernel function can be expressed as a linear combination of Jacobi–Poincaré series of matrix index. Eisenstein and Poincaré series are the most simple examples of modular forms. They are obtained by taking the average of a function over a group (modulo a parabolic subgroup) and hence are invariant under the group action by construction. They satisfy the important property of reproducing Fourier coefficients of cusp forms under a suitably defined scalar product and this is a crucial fact used in [**Bri06**]. Furthermore, while the term "newform" is usually applied to cusp forms, it is important to define this for Eisenstein series as well, in order to obtain a complete description of the spaces of newforms. For this reason, we study Poincaré and Eisenstein series for Jacobi forms of lattice index in Chapter 2.

0.1. Poincaré and Eisenstein series

To the best of the author's knowledge, Poincaré series have not been defined in the literature for Jacobi forms of lattice index. Let $L^{\#}$ denote the dual of L with respect to β and define the following set, which is called the support of $\underline{L}$:

$$\operatorname{supp}(\underline{L}) := \{(D, r) : D \in \mathbb{Q}_{\leq 0}, r \in L^{\#}, D \equiv \beta(r) \bmod \mathbb{Z}\}.$$

For every pair (D, r) in $\operatorname{supp}(\underline{L})$ such that $D < 0$, define the Poincaré series of weight k and index $\underline{L}$ associated with the pair (D, r) as the series

$$P_{k,\underline{L},D,r}(\tau, z) := \sum_{\gamma \in J^{\underline{L}}_{\infty} \backslash J^{\underline{L}}} g_{\underline{L},D,r}|_{k,\underline{L}}\gamma(\tau, z),$$

where $g_{\underline{L},D,r}$ is a simple exponential function, $J^{\underline{L}}$ denotes the integral Jacobi group associated with $\underline{L}$ and $J^{\underline{L}}_{\infty}$ is the stabilizer of the functions $g_{\underline{L},D,r}$ inside $J^{\underline{L}}$. Furthermore, $|_{k,\underline{L}}$ is the action of $J^{\underline{L}}$ on Jacobi forms from Definition 1.22.

In Theorem 2.3, we show that $P_{k,\underline{L},D,r}$ converges absolutely and uniformly on compact subsets of its domain of definition under certain weight restrictions. By computing its Fourier expansion, we show that it is a Jacobi *cusp form* of weight k and index $\underline{L}$. Its Fourier coefficients are expressed in terms of infinite sums containing J-Bessel functions and Gauss-type sums. Furthermore, the series $P_{k,\underline{L},D,r}$ reproduces the Fourier coefficients of Jacobi cusp forms of the same weight and index under the Petersson scalar product defined in (1.21). As a result, the set

$$\{P_{k,\underline{L},D,r} : r \in L^{\#}/L, D \in \mathbb{Q}_{<0} \text{ and } \beta(r) \equiv D \bmod \mathbb{Z}\}$$

generates the $\mathbb{C}$-vector space of Jacobi cusp forms of weight k and index $\underline{L}$. It is well-known that $L^{\#}/L$ is a finite abelian group.

The definition of Jacobi–Eisenstein series of lattice index was given in [**Ajo15**, §3.3], where some of their properties were studied (such as dimension formulas for their spanning set and the fact that they are Hecke eigenforms). These functions are indexed by *isotropic elements*, i.e. elements r in $L^{\#}$ such that $\beta(r) \in \mathbb{Z}$, and they only depend on r modulo L. For every such r, the Eisenstein series of weight k and index $\underline{L}$ associated with r is defined as the series

$$E_{k,\underline{L},r}(\tau, z) := \frac{1}{2} \sum_{\gamma \in J^{\underline{L}}_{\infty} \backslash J^{\underline{L}}} g_{\underline{L},0,r}|_{k,\underline{L}}\gamma(\tau, z).$$

The convergence conditions for $E_{k,\underline{L},r}$ were stated in [**Ajo15**]. In Theorem 2.6, we prove that it is a Jacobi form of weight k and index $\underline{L}$, by computing its Fourier expansion. Its Fourier coefficients are expressed in terms of infinite sums containing Gauss-type sums, as can be seen in (2.16). Furthermore, the series $E_{k,\underline{L},r}$ is orthogonal to cusp forms of the same weight and index under the Petersson scalar product. As a result, we obtain a direct sum decomposition of Jacobi forms with respect to the Petersson scalar product into cusp forms and Eiesenstein series. The proofs of Theorems 2.3 and 2.6 are based on the approach employed in [**BK93**] for the study of Siegel modular forms.

We would like to obtain a closed formula for the Fourier coefficients of Eisenstein series. Lattice sums similar to (2.16) also arise in the Fourier expansions of Poincaré and Eisenstein series for vector-valued modular forms and those of orthogonal modular forms [**BK01, Wil18**], as well as in trace formulas for these types of automorphic forms [**SZ89, Ma18**]. Most of the literature deals with the simplest case, which is equivalent to taking $r = 0$ in $L^\#/L$. Even for Jacobi forms of scalar index, the authors of [**EZ85**] compute the Fourier expansion of $E_{k,m,0}$ and state that "(the calculation) is tedious (for arbitrary r)". The mathematical objects that arise in these calculations are Gauss sums for abelian groups and representation numbers for quadratic forms. For an introduction to these topics, the reader can consult [**Doy16**] and [**Sch85**, §5], respectively, for example. In Lemma 2.10, we show that $E_{k,\underline{L},0}$ vanishes identically when k is odd. In Theorem 2.14, we use results from [**BK01**] on L-series arising from representation numbers of quadratic forms, in order to obtain an explicit formula for the Fourier coefficients of $E_{k,\underline{L},0}$ when k is even. Classical number theoretical objects such as Bernoulli numbers and values of Dirichlet L-functions at negative integers appear in this formula and we show that the final expression is a rational number.

Let N_x denote the order of an element x of $L^\#/L$. For arbitrary isotropic elements r in $L^\#/L$, we use the existence of an isomorphism between spaces of Jacobi forms and spaces of vector-valued modular forms and a linear operator which was defined in [**Wil18**], in order to prove that the Fourier coefficients of

$$\sum_{m \in \mathbb{Z}/N_r\mathbb{Z}} E_{k,\underline{L},mr}$$

are equal to finite linear combinations of Fourier coefficients of $E_{k,\underline{L},0}$. The proof relies heavily on the connection between the Weil and the Schrödinger representations. We use this result to compute the Fourier coefficients of Eisenstein series associated with isotropic elements of small order in Examples 2.26–2.29.

0.2. Hecke operators and the action of the orthogonal group

Hecke operators give extra structure to spaces of automorphic forms and they have algebraic interpretations in terms of the underlying surfaces. They can be used to construct equivariant lifting maps between different types of automorphic forms. Hecke operators acting on Jacobi forms of lattice index were defined in [**Ajo15**, §2.5] as double coset operators (see Definition 3.2). It was shown there that they preserve spaces of Jacobi forms of fixed weight and index and that they are Hermitian under the Petersson scalar product. Their action on the Fourier coefficients of Jacobi forms was computed and their multiplicative properties were studied. Furthermore, by studying the L-functions attached to Hecke eigenforms, a relation between Jacobi forms and elliptic modular forms was formulated. Explicit lifting maps were also defined in some cases and we discuss them in Subsection 3.1.2.

The discriminant module of $\underline{L}$ is the pair $D_{\underline{L}} = (L^\#/L, \beta \bmod \mathbb{Z})$. It is a finite quadratic module (see Definition 1.11). We show in Proposition 3.20 that the orthogonal group of $D_{\underline{L}}$ acts on Jacobi forms of weight k and index $\underline{L}$ from the right. In Proposition 3.22, we prove that the operators arising from the action of the orthogonal group of $D_{\underline{L}}$

are unitary with respect to the Petersson scalar product. In particular, since they commute with Hecke operators and the spaces of Jacobi cusp forms of weight k and index $\underline{L}$ are finite-dimensional, every such space has a basis of common eigenforms. We also compute the action of these operators on Eisenstein series in Proposition 3.24, using the fact that Eisenstein series are uniquely determined by the theta series in their singular terms. Furthermore, reflection maps in the orthogonal group of $D_{\underline{L}}$ act on Jacobi forms as involutions. It is well-known that, in the case of lattices of rank one, reflection maps act on Jacobi forms in the same way that Atkin–Lehner involutions act on elliptic modular forms (see Example 3.28).

The root lattices D_n are defined as

$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \in 2\mathbb{Z}\}.$$

For odd n, the generators for the spaces of Jacobi forms of index D_n over the ring of elliptic modular forms were given in [**BS19**]. In Section 3.3, we use them to compute the Fourier coefficients of Jacobi cusp forms of weight k and index D_n (odd n) for small values of k. We compare their Hecke eigenvalues with the eigenvalues of elliptic modular forms in Table 3.1, in order to verify the conjectured correspondence between Jacobi forms of odd rank lattice index and elliptic modular forms from [**Ajo15**, §6.1.1]. Our calculations suggest that this conjecture is partially correct and we propose a fix for Jacobi forms of index D_n.

0.3. Level raising operators

Level raising operators are intimately connected to the theory of newforms. They can also be used to define additive lifting maps between Jacobi forms and other type of automorphic forms [**CG13, Maa79**].

Level raising operators of type $U(\cdot)$ arise from isometries of lattices (see Definition 4.1). Let $\underline{L_1} = (L_1, \beta_1)$ and $\underline{L_2} = (L_2, \beta_2)$ be positive-definite, even lattices over $\mathbb{Z}$, such that $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$ as modules over $\mathbb{Q}$ and there exists an isometry σ of $\underline{L_1}$ into $\underline{L_2}$. We define a linear operator $U(\sigma)$ and show in Theorem 4.3 that it maps Jacobi forms of weight k and index $\underline{L_2}$ to Jacobi forms of weight k and index $\underline{L_1}$. By analysing Fourier expansions, it is straight-forward to show that such operators preserve cusp forms and Eisenstein series. If $\underline{L_1}$ and $\underline{L_2}$ are as above, then $(\sigma(L_1), \beta_2)$ is a sublattice of $\underline{L_2}$ and $\sigma : \underline{L_1} \to (\sigma(L_1), \beta_2)$ is an isomorphism of lattices. Conversely, every sublattice (M, β_2) of $\underline{L_2}$ gives rise to an isometry of (M, β_2) into $\underline{L_2}$ given by inclusion. In other words, given a positive-definite, even lattice $\underline{L}$, for every overlattice $\underline{L}'$ of $\underline{L}$, Jacobi forms of weight k and index $\underline{L}'$ are Jacobi forms of weight k and index $\underline{L}$. Every Jacobi form of index $\underline{L}'$ is called an oldform of index $\underline{L}$. In Lemma 4.19, we obtain a criterion for when a Jacobi form is an oldform of this type.

Level raising operators of type $V(\cdot)$ were constructed in [**Gri94**] as the images of elliptic Hecke operators under a certain homomorphism of Hecke algebras, using the relation between Jacobi forms and orthogonal modular forms. The reader can also consult Definition 4.25 for a classical approach. In Theorem 4.26, we show that, for every l in $\mathbb{N}$, the operator $V(l)$ maps Jacobi forms of weight k and index $\underline{L} = (L, \beta)$ to Jacobi forms of weight k and index $\underline{L}(l) := (L, l\beta)$ and we compute the action of $V(l)$ on Fourier coefficients of Jacobi forms. As a corollary, the operators $V(\cdot)$ also preserve cusp forms and Eisenstein series. The precise action of $U(\cdot)$ and $V(\cdot)$ on Eisenstein series is given in 4.41. Every Jacobi form ϕ of index $\underline{L}$ gives rise to the oldform $V(l)\phi$ of index $\underline{L}(l)$.

In Section 4.3, we show that $U(\cdot)$ and $V(\cdot)$ commute with each other. They also commute with Hecke operators and with the action of well-defined reflection maps, implying that they are well-suited to develop a theory of newforms for Jacobi forms of lattice index. It was shown in [**Ajo15**, §3.3] that "twisted" Eisenstein series (see Definition 1.31) form a basis of Hecke eigenforms for Hecke operators. We obtain a sufficient condition for twisted Eisenstein series to be oldforms in Theorem 4.39.

CHAPTER 1

Preliminaries

This chapter contains the notation and elementary theory which are necessary in order to make the results in this thesis precise. We recall the definition of Jacobi forms of lattice index, following [**Ajo15**]. We discuss the connection between Jacobi forms and vector-valued modular forms and, finally, we list some examples.

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ denote the set of positive natural numbers, the ring of rational integers, the rational number field, the real number field and the complex number field, respectively. Set $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. The ring of integers modulo n is denoted by $\mathbb{Z}_{(n)}$. For every d in $\mathbb{Z}_{(n)}^\times$, denote the inverse of d modulo n by d^{-1}.

Consider the branch of the complex square root with argument in $(-\pi/2, \pi/2]$. It follows that the function $z \mapsto \sqrt{z}$ takes positive reals to positive reals, complex numbers in the upper half-plane to the first quadrant and complex numbers in the lower half-plane to the fourth quadrant. Set $z^{k/2} := (\sqrt{z})^k$ when $k \in \mathbb{Z}$. Let $\bar{z}$ denote the complex conjugate of a complex number z and let $\Re(z)$ and $\Im(z)$ denote its real and imaginary parts, respectively. For an odd prime p and an integer a, the number $(\frac{a}{p})$ is the usual Legendre symbol and, when $p = 2$, it is equal to 0 when a is even, to 1 when $a \equiv \pm 1 \bmod 8$ and to -1 when $a \equiv \pm 3 \bmod 8$. Define $\left(\frac{a}{1}\right)$ to be equal to 1 and $\left(\frac{a}{-1}\right)$ to be equal to $\mathrm{sign}(a)$. Let n in $\mathbb{Z}$ have prime factorization $up_1^{e_1} p_k^{e_k}$, with $u = \pm 1$. The *Kronecker symbol* $\left(\frac{a}{n}\right)$ is defined as

$$\left(\frac{a}{n}\right) := \left(\frac{a}{u}\right) \prod_{i=1}^{k} (\frac{a}{p_i})^{e_i}.$$

For every prime number p, the p-adic valuation for $\mathbb{Q}$ is defined as

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}, v_p(n) := \begin{cases} \max\{v \in \mathbb{N}, p^v \mid n\}, & \text{if } n \in \mathbb{Z} \setminus \{0\}, \\ v_p(a) - v_p(b), & \text{if } n = \frac{a}{b} \in \mathbb{Q} \setminus \mathbb{Z} \text{ and} \\ \infty, & \text{if } n = 0. \end{cases}$$

The greatest common divisor of two integers a and b is denoted by (a, b). Write $b \parallel a$ if $b \mid a$ and $(b, \frac{a}{b}) = 1$. In sums of the form $\sum_{b|a}$ or $\sum_{ab=n}$, the summation is over positive divisors only. For an integer n, set $e_n(x) := e^{2\pi i x/n}$ and $e^n(x) := e^{2\pi i n x}$. Write $e(x) = e_1(x)$.

The J-Bessel function of index $\alpha > 0$ is defined by the following series expansion around $x = 0$:

$$J_\alpha(x) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n+\alpha}. \tag{1.1}$$

For every c in $\mathbb{N}$ and m, n in $\mathbb{Z} \setminus \{0\}$, define the Kloosterman sum

$$S(m, n; c) := \sum_{a \in \mathbb{Z}_{(c)}^\times} e_c(ma + na^{-1}), \tag{1.2}$$

where a^{-1} denotes the inverse of a modulo c.

Let $\mathbb{Z}_p$ denote the p-adic integers and let $\| \cdot \|_p$ be the p-adic norm on the p-adic numbers, i.e. $\| a \|_p := p^{-v_p(a)}$.

Definition 1.1 (Igusa zeta function). Let $f \in \mathbb{Z}_p[X_1, \ldots, X_e]$. The Igusa zeta function of f at p is defined for every s in $\mathbb{C}$ with $\Re(s) > 0$ as the p-adic integral

$$\zeta(f; p; s) := \int_{\mathbb{Z}_p^e} \| f(x) \|_p^s \, dx.$$

It was proved in [**Igu74**] that $\zeta(f; p; s)$ is a rational function in p^{-s} and hence it has a meromorphic continuation to all of $\mathbb{C}$.

Let $\omega(\cdot)$, $\mu(\cdot)$, $\sigma_t(\cdot)$ and $\zeta(\cdot)$ denote the function counting the number of prime divisors of an integer, the Möbius function, the t-th divisor sum and the Riemann zeta function, respectively. We define $\sigma_t(n) = 0$ for n in $\mathbb{R} \setminus \mathbb{N}$. Let B_n denote the n-th Bernoulli number and define the n-th Bernoulli polynomial

$$B_n(x) := \sum_{j=0}^{n} \binom{n}{j} B_{n-j} x^j. \tag{1.3}$$

We remind the reader of the following well-known identity:

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & n = 1 \text{ and} \\ 0, & \text{otherwise.} \end{cases} \tag{1.4}$$

Let R be a ring. The set of $n \times n$ matrices with entries in R is denoted by $\mathrm{M}_n(R)$. A matrix A in $\mathrm{M}_n(\mathbb{Z})$ is called even if it has even diagonal entries. Denote the group of invertible matrices in $\mathrm{M}_n(R)$ by $\mathrm{GL}_n(R)$ and the group of matrices with determinant equal to one by $\mathrm{SL}_n(R)$. If $R \subseteq \mathbb{R}$, then denote the group of matrices with positive determinant by $\mathrm{GL}_n^+(R)$. For every $n \times m$ matrix A, its transpose is denoted by A^t.

Let $N \geq 1$ be an integer. A *Dirichlet character modulo N* is a map $\chi : \mathbb{Z} \to \mathbb{C}$ which satisfies the following properties:

- $\chi(x + N) = \chi(x)$ for all x in $\mathbb{Z}$,
- $\chi(x) = 0$ if and only if $(x, N) > 1$,
- $\chi(xy) = \chi(x)\chi(y)$ for all x, y in $\mathbb{Z}$.

For every Dirichlet character χ, let σ_t^χ denote the twisted divisor sum

$$\sigma_t^\chi(n) := \sum_{d|n} \chi(d) d^t$$

and, for every two Dirichlet characters ξ and χ, set

$$\sigma_t^{\xi,\chi}(n) := \sum_{d|n} \xi\left(\frac{n}{d}\right) \chi(d) d^t.$$

The *Dirichlet L-function* of a Dirichlet character χ is

$$L(s, \chi) := \sum_{n=1}^{\infty} \chi(n) n^{-s} = \prod_p \left(1 - \chi(p) p^{-s}\right)^{-1}.$$

For every positive integer N, set

$$L_N(s, \chi) := \sum_{\substack{n=1 \\ (n,N)=1}}^{\infty} \chi(n) n^{-s} = L(s, \chi) \prod_{p|N} \left(1 - \chi(p) p^{-s}\right).$$

A *discriminant* is an integer which is congruent to 0 or 1 modulo 4. For every discriminant D, the function $\chi_D := \left(\frac{D}{\cdot}\right)$ is a well-defined quadratic Dirichlet character and we set $L_D(\cdot) := L(\cdot, \chi_D)$. A *fundamental discriminant* is an integer d such that either $d \equiv 1 \bmod 4$ and d is square-free or $d = 4n$ for some n in $\mathbb{Z}$ such that $n \equiv 2$ or $3 \bmod 4$ and n is square-free.

DEFINITION 1.2 (Conductor). Let ξ and χ be two Dirichlet characters modulo F and N, respectively, with $F \mid N$. If $\chi(n) = \xi(n)$ for every n in $\mathbb{Z}_{(N)}^{\times}$, then χ is *inflated* by ξ. If χ is not inflated by any character other than itself, then it is called *primitive*. It is well-known that every Dirichlet character χ is induced by a primitive Dirichlet character which is uniquely determined by χ. The conductor of χ is the period of the primitive character which induces it.

Let χ be a primitive character modulo N and define the Gauss sum

$$G(\chi) := \sum_{n=1}^{N} \chi(n) e_N(n)$$

and the constant

$$a_\chi = \begin{cases} 0, & \text{if } \chi(-1) = 1 \text{ and} \\ 1, & \text{if } \chi(-1) = -1. \end{cases}$$

Define the *completed L-function* of χ as

$$\Lambda(s,\chi) := \left(\frac{N}{\pi}\right)^{\frac{s+a_\chi}{2}} \Gamma\left(\frac{s+a_\chi}{2}\right) L(s,\chi).$$

The following holds:

$$\Lambda(1-s,\chi) = \frac{G(\chi)}{((-1)^{a_\chi} N)^{\frac{1}{2}}} \Lambda(s,\overline{\chi}). \tag{1.5}$$

For a proof of this fact, the reader can consult [**CS17**, §3.4.3], for example.

1.1. Modular forms

Let $\mathfrak{H}$ denote the *upper half-plane*

$$\{z \in \mathbb{C} : \Im(z) > 0\}.$$

For every τ in $\mathfrak{H}$ and z in $\mathbb{C}$, write q for $e^{2\pi i\tau}$ and ζ for $e^{2\pi i z}$. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathfrak{H}$ via linear fractional transformations:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d}.$$

For every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{GL}_2^+(\mathbb{R})$ and every τ in $\mathfrak{H}$, define the automorphy factor $j(A,\tau) := c\tau + d$. For every integer k, define a right-action of $\mathrm{GL}_2^+(\mathbb{Q})$ on the space of functions $f : \mathfrak{H} \to \mathbb{C}$ in the following way:

$$(f, A) \mapsto (f|_k A)(\tau) := \det(A)^{\frac{k}{2}} j(A,\tau)^{-k} f(A\tau).$$

Let Γ denote the *modular group* $\mathrm{SL}_2(\mathbb{Z})$ and, for every positive integer N, set

$$\Gamma(N) := \left\{ A \in \Gamma : A \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\} \text{ and}$$
$$\Gamma_0(N) := \left\{ A \in \Gamma : A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \bmod N \right\}.$$

A *congruence subgroup* of Γ is a subgroup containing $\Gamma(N)$ for some N. The smallest possible such N is called the *level* of the congruence subgroup. A *cusp* of a congruence subgroup G is an equivalence class of $\mathbb{P}^1(\mathbb{Q})$ under the action of G and a representative of such an equivalence class is also called a cusp. Let Γ_∞ denote the stabilizer of the cusp $(i\infty)$ in Γ, i.e. the subgroup $\{\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) : n \in \mathbb{Z}\}$ of Γ.

If $k \in \mathbb{Z}$, then a *multiplier system* of weight k for G is a homomorphism $v : G \to S^1$ if $k \in \mathbb{Z}$. If $k \in \mathbb{Z} + \frac{1}{2}$, then a multiplier of weight k for G is a function $v : G \to$

S^1 such that $v(g_1)v(g_2) = \sigma(g_1, g_2)v(g_1g_2)$ for every g_1, g_2 in G, where $\sigma(g_1, g_2) = j(g_1, g_2\tau)^{1/2} j(g_2, \tau)^{1/2} j(g_1g_2, \tau)^{-1/2} \in \{\pm 1\}$ is independent of τ. In either case, v must additionally satisfy $v(-I_2) = e^{-\pi i k}$ if $-I_2 \in G$.

Let $k \in \mathbb{Z}$, $N \in \mathbb{N}$, let G be a congruence subgroup of level N and let v be a multiplier system of weight k for G. An *elliptic modular form of weight k with multiplier system v for G* is a holomorphic function $f : \mathfrak{H} \to \mathbb{C}$ which satisfies the following properties:

- $f|_k A = v(A) f$ for every A in G,
- the function f is holomorphic at the cusps of G.

Every f as above has a Fourier expansion of the form

$$f(\tau) = \sum_{n \geq 0} a_f(n) q^{n/w},$$

where w is the *width* of the cusp $i\infty$ [**CS17**, §7.1]. The elliptic modular form f is called a *cusp form* if it vanishes at the cusps of G. The $\mathbb{C}$-vector space of elliptic modular forms of weight k with trivial multiplier system for $\Gamma_0(N)$ is denoted by $M_k(N)$ and its subspace of cusp forms is denoted by $S_k(N)$. If χ is a Dirichlet character modulo N and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$, then set $\chi(A) := \chi(d)$. The map $A \mapsto \chi(A)$ defines a multiplier system of even integral weight for $\Gamma_0(N)$, which we denote by the same symbol χ. Denote the $\mathbb{C}$-vector space of elliptic modular forms of weight k with character χ for $\Gamma_0(N)$ by $M_k(N, \chi)$ and its subspace of cusp forms by $S_k(N, \chi)$.

It is also possible to define elliptic modular forms of *half-integral* weight, whose theory was established by Shimura [**Shi73**]. For example, the Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n \geq 1} (1 - q^n) = \frac{1}{2} \sum_{n \in \mathbb{Z}} \left(\frac{12}{n}\right) q^{\frac{n^2}{24}} \tag{1.6}$$

is a modular form of weight $1/2$ for Γ with multiplier system of order 24 given by

$$v_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{|c|}\right) \exp\left(\frac{\pi}{12}((a + d - 3)c - bd(c^2 - 1))\right), & \text{if } 2 \nmid c \text{ and} \\ \left(\frac{c}{|d|}\right) \exp\left(\frac{\pi}{12}((a - 2d)c - bd(c^2 - 1) + 3d - 3)\right) \varepsilon(c, d), & \text{if } 2 \mid c, \end{cases}$$

where

$$\varepsilon(c, d) := \begin{cases} -1, & \text{if } c \leq 0 \text{ and } d < 0 \text{ and} \\ 1, & \text{otherwise.} \end{cases}$$

Together with the *scalar Jacobi theta series*, the Dedekind η-function can be used as a building block for Jacobi forms, as we shall see in Subsection 1.3.3.

For every l in $\mathbb{N}$, define the following operators on $M_k(N, \chi)$:

$$\begin{aligned} U(l) f(\tau) &:= \sum_{n \geq 0} a_f(ln) q^n, \\ V(l) f(\tau) &:= \sum_{n \geq 0} a_f(n) q^{ln} \text{ and} \\ T(l) f(\tau) &:= l^{\frac{k}{2} - 1} \sum_{ad = l} \sum_{b \bmod d} \chi(a) f|_k \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)(\tau). \end{aligned} \tag{1.7}$$

It is well-known that the Hecke operators $T(\cdot)$ map $M_k(N, \chi)$ to itself and that $U(l)$ and $V(l)$ map $M_k(N, \chi)$ to $M_k(lN, \chi)$ (see [**DS05**, §5], for example). Furthermore, if $l \mid N$, then $U(l) f$ is an element of $M_k(N, \chi)$.

Let $f = \sum_n a_f(n) q^n$ be an elliptic modular form in $M_k(N, \chi)$, which is a normalized eigenfunction of the Hecke operators $T(l)$ for all l in $\mathbb{N}$. The L-series of f in s is defined as

$$L(s,f)=\sum_{n=1}^{\infty}a_f(n)n^{-s}.$$

It has an Euler product of the form

$$L(s,f)=\prod_p\left(1-a_f(p)p^{-s}+\chi(p)p^{k-1-2s}\right)^{-1}. \tag{1.8}$$

The reader can consult [**CS17**, §10.7] for a proof of this fact when $f\in M_k(1)$ and the same argument holds for f in $M_k(N,\chi)$. Define the *completed L-function* of f as

$$\Lambda_N(s,f):=\left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\Gamma(s)L(s,f).$$

Definition 1.3 (Metaplectic group). The metaplectic group, denoted by $\tilde{\Gamma}$, consists of pairs $\tilde{A}:=(A,w(\tau))$ with A in Γ and $w:\mathfrak{H}\to\mathbb{C}$ a holomorphic function satisfying $w(\tau)^2=j(A,\tau)$. The group law on $\tilde{\Gamma}$ is

$$(A,w(\tau))(B,v(\tau))=(AB,w(B\tau)v(\tau)).$$

The metaplectic group is a double cover of Γ and it is generated by the following elements:

$$\tilde{T}=\left(\begin{pmatrix}1&1\\0&1\end{pmatrix},1\right)\qquad\text{and}\qquad\tilde{S}=\left(\begin{pmatrix}0&-1\\1&0\end{pmatrix},\sqrt{\tau}\right).$$

Definition 1.4 (Vector-valued modular forms). Let V be a finite-dimensional vector space over $\mathbb{C}$. For every half-integer k, define a right-action of $\tilde{\Gamma}$ on the space of functions $F:\mathfrak{H}\to V$ in the following way:

$$(F,\tilde{A})\mapsto F|_k\tilde{A}(\tau):=w(\tau)^{-2k}F(A\tau).$$

Let $\rho:\tilde{\Gamma}\to\mathrm{Aut}(V)$ be a finite-dimensional representation of $\tilde{\Gamma}$, whose kernel has finite index in $\tilde{\Gamma}$. A vector-valued modular form of weight k for ρ is a holomorphic function $F:\mathfrak{H}\to V$ which satisfies

$$F|_k\tilde{A}(\tau)=\rho(\tilde{A})F(\tau)$$

for every element $\tilde{A}$ of $\tilde{\Gamma}$. Denote the $\mathbb{C}$-vector space of all such functions by $M_k(\rho)$.

Let I_n denote the $n\times n$ identity matrix and set $E_n=\left(\begin{smallmatrix}0&I_n\\-I_n&0\end{smallmatrix}\right)$. The *symplectic group* $\mathrm{Sp}_n(\mathbb{R})$ is the set of $2n\times 2n$ matrices M in $\mathrm{GL}_n(\mathbb{R})$ satisfying $M^TE_nM=E_n$. We often consider its subgroup $\mathrm{Sp}_n(\mathbb{Z})$ of matrices with integer entries. The *Siegel upper half-space of degree n*, denoted by $\mathfrak{H}_n$, is the set of complex, symmetric $n\times n$ matrices with positive-definite imaginary part. The group $\mathrm{Sp}_n(\mathbb{R})$ acts on $\mathfrak{H}_n$ via

$$\left(\begin{pmatrix}A&B\\C&D\end{pmatrix},Z\right)\mapsto\begin{pmatrix}A&B\\C&D\end{pmatrix}Z:=(AZ+B)(CZ+D)^{-1}.$$

Let $k\in\mathbb{Z}$ and $n\in\mathbb{N}$ such that $n>1$ and let G be a subgroup of $\mathrm{Sp}_n(\mathbb{Z})$. A *Siegel modular form of weight k and degree n for G* is a holomorphic function $F:\mathfrak{H}_n\to\mathbb{C}$ which satisfies

$$F\left(\begin{pmatrix}A&B\\C&D\end{pmatrix}Z\right)=\det(CZ+D)^kF(Z)$$

for every $\left(\begin{smallmatrix}A&B\\C&D\end{smallmatrix}\right)$ in G. An analogous definition can be given for every finite index subgroup of $\mathrm{Sp}_n(\mathbb{Z})$.

1.2. Lattices

Let R be a commutative ring and let L and N be R-modules, with L free of finite rank equal to g. A map $\beta : L \times L \to N$ is called a *symmetric R-bilinear form* if

$$\beta(x, y) = \beta(y, x) \qquad \text{and} \qquad \beta(x, my + nz) = m\beta(x, y) + n\beta(x, z)$$

for all x, y, z in L and all m, n in R. If $N = R$, then β is called *integral*. If $\beta(x, y) = 0$ for all y in L if and only if $x = 0$, then β is called *non-degenerate*. Let $\{e_1, \dots, e_g\}$ be an R-basis of L. The matrix $G = (\beta(e_i, e_j))_{i,j}$ is called the *Gram matrix* of β with respect to $\{e_1, \dots, e_g\}$. Let $\tilde{x}$ and $\tilde{y}$ be the column vectors whose entries are the coefficients of x and y with respect to $\{e_1, \dots, e_g\}$. Then

$$\beta(x, y) = \sum_{i=1}^{g} \sum_{j=1}^{g} \tilde{x}_i \tilde{y}_j \beta(e_i, e_j) = \tilde{x}^t G \tilde{y}. \tag{1.9}$$

Definition 1.5 (Lattice). Let L and N be R-modules, with L free of finite rank, and let $\beta : L \times L \to N$ be a symmetric, non-degenerate bilinear form. The pair $\underline{L} = (L, \beta)$ is called a lattice over R.

The lattice $\underline{L}$ is called *integral* if the associated bilinear form is integral. By abuse of notation, denote the quadratic form associated with $\underline{L}$ by $\beta(\cdot)$, i.e.

$$\beta(x) := \frac{1}{2}\beta(x, x).$$

Throughout this thesis, we consider only $R = \mathbb{Z}$. Given an arbitrary $\mathbb{Z}$-basis of L, identify every element in the lattice with its coefficient vector and drop the tilde from the notation, i.e. write $\beta(x, y) = x^t G y$. Using the matrix formula (1.9), it is possible to extend the domain of definition of β to $L \otimes_{\mathbb{Z}} \mathbb{Q}$, $L \otimes_{\mathbb{Z}} \mathbb{R}$ and $L \otimes_{\mathbb{Z}} \mathbb{C}$ in a natural way. For every $z = (z_1, \dots, z_{\mathrm{rk}(\underline{L})})$ in $L \otimes_{\mathbb{Z}} \mathbb{C}$, let $\mathfrak{R}(z) = (\mathfrak{R}(z_1), \dots, \mathfrak{R}(z_{\mathrm{rk}(\underline{L})}))$ and $\mathfrak{I}(z) = (\mathfrak{I}(z_1), \dots, \mathfrak{I}(z_{\mathrm{rk}(\underline{L})}))$ denote its real and imaginary parts, respectively.

An integral lattice $\underline{L} = (L, \beta)$ is called *positive-definite* if $\beta(x, x) > 0$ for all x in L such that $x \neq 0$. It is called *even* if $\beta(x, x)$ is even for all x in L, otherwise it is called *odd*. The *rank* of $\underline{L} = (L, \beta)$, denoted by $\mathrm{rk}(\underline{L})$, is defined as the rank of L as a $\mathbb{Z}$-module.

Example 1.6. The following are examples of positive-definite, even lattices over $\mathbb{Z}$:

(1) For every positive integer m, the lattice $\underline{L}_m := (\mathbb{Z}, (x, y) \mapsto 2mxy)$.
(2) More generally, for every positive-definite, even, $g \times g$ matrix G, the lattice $\underline{L}_G := (\mathbb{Z}^g, (x, y) \mapsto x^t G y)$.
(3) For every positive integer n, the $\mathbb{Z}$-module

$$D_n = \{(x_1, x_2, \dots, x_n) \subseteq \mathbb{Z}^n : x_1 + \dots + x_n \in 2\mathbb{Z}\},$$

equipped with the Euclidean bilinear form

$$(x_1, \dots, x_n)(y_1, \dots, y_n) \mapsto x_1 y_1 + \dots + x_n y_n.$$

(4) For every positive integer n, the $\mathbb{Z}$-module

$$A_n = \left\{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{Z}^{n+1} : x_1 + \dots + x_{n+1} = 0\right\},$$

equipped with the Euclidean bilinear form.
(5) The $\mathbb{Z}$-module

$$E_8 = \left\{(x_1, x_2, \dots, x_8) : \text{ all } x_i \in \mathbb{Z} \text{ or all } x_i \in \mathbb{Z} + \tfrac{1}{2}, x_1 + \dots + x_8 \in 2\mathbb{Z}\right\},$$

equipped with the Euclidean bilinear form.

Definition 1.7. For every lattice $\underline{L} = (L, \beta)$ and every m in $\mathbb{Z}$, set $\underline{L}(m) := (L, m\beta)$.

If M is a free sub-module of L of finite rank equal to $\mathrm{rk}(\underline{L})$ and M has finite index in L, then (M,β) is called a *sublattice* of $\underline{L}$ and $\underline{L}$ is called an *overlattice* of (M,β). Two lattices $\underline{L}_1 = (L_1,\beta_1)$ and $\underline{L}_2 = (L_2,\beta_2)$ are *isomorphic* if there exists an isomorphism of underlying $\mathbb{Z}$-modules $\sigma : L_1 \xrightarrow{\sim} L_2$ such that $\beta_1 = \beta_2 \circ \sigma$. The isomorphisms between $\underline{L}$ and itself form the *orthogonal group of* $\underline{L}$, denoted by $O(\underline{L})$.

For the remainder of this section, assume that $\underline{L} = (L,\beta)$ is an even lattice over $\mathbb{Z}$. Define the following $\mathbb{Z}$-module:

$$L^{\#} = \{y \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(y,x) \in \mathbb{Z} \ \forall \ x \text{ in } L\}.$$

The *dual lattice* of $\underline{L}$ is the pair $\underline{L}^{\#} = (L^{\#},\beta)$. It is well-known that, if $\underline{L}$ has Gram matrix G with respect to some basis $\{e_1,\dots,e_{\mathrm{rk}(\underline{L})}\}$ of L, then a $\mathbb{Z}$-basis of $L^{\#}$ is given by the *dual basis* $\{e_1^{\#},\dots,e^{\#}_{\mathrm{rk}(\underline{L})}\}$, where

$$e_i^{\#} = \sum_{j=1}^{\mathrm{rk}(\underline{L})} G^{-1}_{ji} e_j$$

and the Gram matrix of $\underline{L}^{\#}$ with respect to this basis is equal to G^{-1}.

An integral lattice $\underline{L} = (L,\beta)$ is called *unimodular* if $L^{\#} = L$. For example, the lattice E_8 from Example 1.6, (5) is unimodular.

If $\{f_1,\dots,f_{\mathrm{rk}(\underline{L})}\}$ is another $\mathbb{Z}$-basis of L, then consider the change of coordinates map

$$U : L \to L, U(e_i) = \sum_{j=1}^{\mathrm{rk}(\underline{L})} U_{ji} f_j.$$

Its matrix $U = (U_{ij})_{i,j}$ is an element of $\mathrm{GL}_{\mathrm{rk}(\underline{L})}(\mathbb{Z})$ and, if $\overline{x}$ is the column vector whose entries are the coefficients of x with respect to the new basis, then $U\tilde{x} = \overline{x}$ and the Gram matrix of $\underline{L}$ with respect to $\{f_1,\dots,f_{\mathrm{rk}(\underline{L})}\}$ is equal to $G' = (U^{-1})^t G U^{-1}$. Let G be the Gram matrix of $\underline{L}$ with respect to some basis of L. The *determinant* of $\underline{L}$ is defined as $\det(\underline{L}) := |\det(G)|$. The previous discussion implies that this quantity is independent of change of basis. It is well-known that $L^{\#}/L$ is a finite abelian group of order equal to $\det(\underline{L})$.

The *level* of $\underline{L}$, denoted by $\mathrm{lev}(\underline{L})$, is the smallest positive integer which satisfies $\mathrm{lev}(\underline{L})\beta(x) \in \mathbb{Z}$ for all x in $L^{\#}$. It is well-known that $\mathrm{lev}(\underline{L})$ is the smallest positive integer such that $\mathrm{lev}(\underline{L})G^{-1}$ is an even matrix, independent of the choice of basis for $\underline{L}$ [**Ebe13**, §3.1]. The following remark from [**CS17**, §14.3] plays an important role throughout this thesis:

Remark 1.8. If $\underline{L}$ is even, then $\mathrm{lev}(\underline{L})L^{\#} \subseteq L$. Furthermore, the level and the discriminant of $\underline{L}$ have the same set of prime divisors: if $\mathrm{rk}(\underline{L})$ is even, then $\mathrm{lev}(\underline{L}) \mid \det(\underline{L}) \mid \mathrm{lev}(\underline{L})^{\mathrm{rk}(\underline{L})}$ and, if $\mathrm{rk}(\underline{L})$ is odd, then $2 \mid \det(\underline{L})$ and $4 \mid \mathrm{lev}(\underline{L}) \mid 2\det(\underline{L}) \mid \mathrm{lev}(\underline{L})^{\mathrm{rk}(\underline{L})}$.

Definition 1.9. Set

$$\Delta(\underline{L}) := \begin{cases} (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \det(\underline{L}), & \text{if } \mathrm{rk}(\underline{L}) \equiv 0 \bmod 2 \text{ and} \\ (-1)^{\lfloor \frac{\mathrm{rk}(\underline{L})}{2} \rfloor} 2\det(\underline{L}), & \text{if } \mathrm{rk}(\underline{L}) \equiv 1 \bmod 2. \end{cases}$$

It is well-known that $\Delta(\underline{L})$ is a discriminant (see Lemma 14.3.20 and Remark 14.3.23 in [**CS17**, §14.3]).

Definition 1.10. For every a in $\mathbb{N}$ and every D in $\mathbb{Q}$ such that $D\Delta(\underline{L}) \in \mathbb{Z}$, set

$$\chi_{\underline{L}}(D,a) := \left(\frac{D\cdot\Delta(\underline{L})}{a}\right) \qquad \text{and} \qquad \chi_{\underline{L}}(a) := \chi_{\underline{L}}(1,a).$$

Since $\Delta(\underline{L})$ is a discriminant, the function $\chi_{\underline{L}}(\cdot)$ is a well-defined quadratic character modulo $|\Delta(\underline{L})|$.

Definition 1.11 (Finite quadratic module). A finite quadratic module over $\mathbb{Z}$ is a pair (M, Q), such that M is an abelian group of finite order and $Q : M \to \mathbb{Q}/\mathbb{Z}$ is a non-degenerate quadratic form on M, i.e

- $Q(ax) = a^2 Q(x)$ for all a in $\mathbb{Z}$ and all x in M,
- the symmetric form $\beta : M \times M \to \mathbb{Q}/\mathbb{Z}$ defined by
$$\beta(x, y) = Q(x + y) - Q(x) - Q(y)$$
is $\mathbb{Z}$-bilinear and non-degenerate.

Theorem 1.12 ([**Sko19**, Thm 1.1.8]). *Every finite quadratic module (M, Q) is isomorphic to a direct sum of finite quadratic modules of the following type (called Jordan constituents):*

- $A^t_{p^n} := (\mathbb{Z}_{(p^n)}, r \mapsto \frac{tr^2}{p^n} + \mathbb{Z})$*, for some odd prime p and some integer t such that* $(t, p) = 1$,
- $A^t_{2^n} := (\mathbb{Z}_{(2^n)}, r \mapsto \frac{tr^2}{2^{n+1}} + \mathbb{Z})$*, for some odd integer t,*
- $B_{2^n} := \left(\mathbb{Z}_{(2^n)} \times \mathbb{Z}_{(2^n)}, (r, s) \mapsto \frac{r^2+rs+s^2}{2^n} + \mathbb{Z}\right)$,
- $C_{2^n} := \left(\mathbb{Z}_{(2^n)} \times \mathbb{Z}_{(2^n)}, (r, s) \mapsto \frac{rs}{2^n} + \mathbb{Z}\right)$.

Definition 1.13 (Discriminant module). When $\underline{L}$ is even, the reduction of β modulo $\mathbb{Z}$ induces a bilinear form on $L^\#/L$. The discriminant module associated with $\underline{L}$ is the pair
$$D_{\underline{L}} := \left(L^\#/L, x + L \mapsto \beta(x) + \mathbb{Z}\right).$$
It is a finite quadratic module over $\mathbb{Z}$.

The orthogonal group of $D_{\underline{L}}$, denoted by $O(D_{\underline{L}})$, consists of all group automorphisms α of $L^\#/L$ such that $\beta \circ \alpha = \beta$. Every automorphism of $\underline{L}$ extends to an automorphism of $\underline{L}^\#$, which in turn induces an automorphism of $D_{\underline{L}}$. Hence, there is an induced homomorphism between $O(\underline{L})$ and $O(D_{\underline{L}})$ (which need not be injective or surjective).

For every element x in $L^\#$, let N_x denote the order of $x + L$ in $L^\#/L$, i.e. the smallest positive integer such that $N_x x \in L$. Let $\operatorname{lev}(x)$ denote the smallest positive integer such that $\operatorname{lev}(x)\beta(x) \in \mathbb{Z}$.

Remark 1.14. Since $\beta(x, N_x x) \in \mathbb{Z}$ and $\beta(N_x x, N_x x) \in 2\mathbb{Z}$ for every x in $L^\#$, it follows that $\operatorname{lev}(x) \mid 2N_x$ and that $\operatorname{lev}(x) \mid N_x^2$. In particular, we have $\operatorname{lev}(x) \mid N_x$ when N_x is odd.

The *isotropy set* of $D_{\underline{L}}$ is
$$\operatorname{Iso}(D_{\underline{L}}) := \{x \in D_{\underline{L}} : \beta(x) = 0\}.$$
Let $\mathcal{I}_{\underline{L}}$ denote the set of isotropic subgroups of $D_{\underline{L}}$.

Definition 1.15. There is an action of $\mathbb{Z}^\times_{(\operatorname{lev}(\underline{L}))}$ on $\operatorname{Iso}(D_{\underline{L}})$ given by right multiplication. Let $\mathscr{R}_{\mathrm{Iso}}$ be a set of representatives of the orbit space $\operatorname{Iso}(D_{\underline{L}})/\mathbb{Z}^\times_{(\operatorname{lev}(\underline{L}))}$. Note that $N_x = N_r$ if $x = r$ in $\mathscr{R}_{\mathrm{Iso}}$.

Consider the group algebra $\mathbb{C}[L^\#/L]$ of maps $L^\#/L \to \mathbb{C}$, with natural basis $\{\mathfrak{e}_x\}_{x \in L^\#/L}$. Define a scalar product on $\mathbb{C}[L^\#/L]$ as
$$\left\langle \sum_{x \in L^\#/L} f_x \mathfrak{e}_x, \sum_{x \in L^\#/L} g_x \mathfrak{e}_x \right\rangle := \sum_{x \in L^\#/L} f_x \overline{g_x}.$$

Definition 1.16 (Weil representation). The Weil representation associated with $\underline{L}$ of $\tilde{\Gamma}$ on $\mathrm{Aut}(\mathbb{C}[L^\#/L])$ is defined by the following action of the generators of $\tilde{\Gamma}$ on the basis elements of $\mathbb{C}[L^\#/L]$:

$$\begin{aligned}\rho_{\underline{L}}(\tilde{T})\mathfrak{e}_x &= e(\beta(x))\mathfrak{e}_x,\\ \rho_{\underline{L}}(\tilde{S})\mathfrak{e}_x &= \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}}}\sum_{y\in L^\#/L} e(-\beta(x,y))\mathfrak{e}_y.\end{aligned}$$

In general, write

$$\rho_{\underline{L}}(\tilde{A})\mathfrak{e}_y = \sum_{x\in L^\#/L}\rho_{\underline{L}}(\tilde{A})_{x,y}\mathfrak{e}_x$$

for every element $\tilde{A}$ of $\tilde{\Gamma}$. It is well-known that $\rho_{\underline{L}}$ is unitary and hence its dual representation is given by the formula

$$\rho^*_{\underline{L}}(\tilde{A})\mathfrak{e}_y = \sum_{x\in L^\#/L}\overline{\rho_{\underline{L}}(\tilde{A})_{x,y}}\mathfrak{e}_x.$$

Definition 1.17 (Direct sum of two lattices). Let $\underline{L}_1 = (L_1,\beta_1)$ and $\underline{L}_2 = (L_2,\beta_2)$ be two even lattices and define a symmetric, non-degenerate bilinear form on $L_1 \oplus L_2$ as $f : (L_1 \oplus L_2) \times (L_1 \oplus L_2) \to \mathbb{Z}$,

$$f(x_1 \oplus x_2, y_1 \oplus y_2) := \beta_1(x_1, y_1) + \beta_2(x_2, y_2).$$

The direct sum of $\underline{L}_1$ and $\underline{L}_2$ is the even lattice $\underline{L}_1 \oplus \underline{L}_2 := (L_1 \oplus L_2, f)$.

Definition 1.18 (Stably isomorphic lattices). Two even lattices $\underline{L}_1$ and $\underline{L}_2$ are stably isomorphic if and only if there exist even unimodular lattices $\underline{U}_1$ and $\underline{U}_2$ such that $\underline{L}_1 \oplus \underline{U}_1 \simeq \underline{L}_2 \oplus \underline{U}_2$.

Theorem ([**Nik80**, Thm 1.3.1]). *Two even integral lattices are stably isomorphic if and only if their discriminant modules are isomorphic.*

Let F be a field of characteristic different from two. A *quadratic space* over F is a pair (V, Q), such that V is a finite-dimensional F-module and $Q : V \to F$ is a quadratic form on V. Let (V_1, Q_1) and (V_2, Q_2) be two quadratic spaces over F. A *representation of V_1 into V_2 with respect to Q_1 and Q_2* is a linear map $\sigma : V_1 \to V_2$ which satisfies

$$Q_2 \circ \sigma(x) = Q_1(x), \quad \text{for all } x \text{ in } V_1.$$

When $F = \mathbb{Q}$, every such function can be extended to a function $\sigma : V_1 \otimes_{\mathbb{Z}} \mathbb{C} \to V_2 \otimes_{\mathbb{Z}} \mathbb{C}$ by linearity. If β_1 and β_2 denote the bilinear forms associated with Q_1 and Q_2, respectively, then every representation σ of V_1 into V_2 satisfies

$$\beta_2(\sigma(x), \sigma(y)) = \beta_1(x, y) \quad \text{for all } x, y \text{ in } V_1.$$

An *isometry* of (V_1, Q_1) into (V_2, Q_2) is an injective representation of V_1 into V_2 with respect to Q_1 and Q_2.

Definition 1.19 (Isometry of lattices). Let $\underline{L}_1$ and $\underline{L}_2$ be lattices in (V_1, Q_1) and (V_2, Q_2), respectively. An isometry of $\underline{L}_1$ into $\underline{L}_2$ is an isometry σ of (V_1, Q_1) into (V_2, Q_2), such that $\sigma\underline{L}_1 \subseteq \underline{L}_2$.

Fix any two $\mathbb{Z}$-bases of L_1 and L_2 and let G_1 and G_2 denote the Gram matrices of $\underline{L}_1$ and $\underline{L}_2$, respectively. Let M denote the matrix of σ with respect to these bases. The relation $Q_2 \circ \sigma = Q_1$ implies that

$$M^t G_2 M = G_1.$$

Hence, if T and U are change of coordinates maps for $\underline{L}_1$ and $\underline{L}_2$, respectively, then the matrix of σ with respect to the new bases is equal to UMT^{-1}. When $\mathrm{rk}(\underline{L}_1) = \mathrm{rk}(\underline{L}_2)$, set $\det(\sigma) := |\det(M)|$.

1.3. Jacobi modular forms

For the remainder of this chapter, assume that $\underline{L} = (L, \beta)$ is a positive-definite, even lattice over $\mathbb{Z}$. In order to define the Jacobi group, we first need to define the Heisenberg group. This group originates from quantum mechanics, more precisely in the description of one-dimensional mechanical systems. In number theory, it is intimately related to theta series via its *theta representation*. For details on this topic, see [**Mum07**, §1.3]. We follow the exposition in [**Ajo15**] and the reader can consult the cited text for details and proofs.

Definition 1.20 (Heisenberg group). The Heisenberg group associated with $\underline{L}$ is the set

$$H^{\underline{L}}(\mathbb{R}) := \{(x, y, \zeta) : x, y \in L \otimes \mathbb{R}, \zeta \in S^1\},$$

equipped with the following composition law:

$$(x_1, y_1, \zeta_1)(x_2, y_2, \zeta_2) := (x_1 + x_2, y_1 + y_2, \zeta_1\zeta_2 e\left(\beta(x_1, y_2)\right)).$$

The *integral Heisenberg group* is the subgroup $H^{\underline{L}}(\mathbb{Z}) := \{(x, y, 1) : x, y \in L\}$ of $H^{\underline{L}}(\mathbb{R})$. Drop the third entry from the notation for this group for simplicity. This group is sometimes called the *reduced* Heisenberg group in the literature.

Proposition ([**Ajo15**, Prop 2.2.3]). *The group* $\mathrm{SL}_2(\mathbb{R})$ *acts on* $H^{\underline{L}}(\mathbb{R})$ *from the right via*

$$((x, y, \zeta), A) \mapsto (x, y, \zeta)^A := ((x, y)A, \zeta e_2\left(\beta\left((x, y)A\right) - \beta(x, y)\right)),$$

where $(x, y)A$ *is the vector obtained by multiplying the row vector* (x, y) *with* A.

Definition 1.21 (Jacobi group). The real Jacobi group associated with $\underline{L}$, denoted by $J^{\underline{L}}(\mathbb{R})$, is the semi-direct product of $\mathrm{SL}_2(\mathbb{R})$ and $H^{\underline{L}}(\mathbb{R})$. The composition law on this group is

$$(A, h) \cdot (A', h') = (AA', h^{A'} h').$$

The following holds:

Proposition ([**Ajo15**, Prop 2.2.7]). *The real Jacobi group acts on the left on the space* $\mathfrak{H} \times (L \otimes \mathbb{C})$*: if* $A \in \mathrm{SL}_2(\mathbb{R})$ *and* $h = (x, y, \zeta) \in H^{\underline{L}}(\mathbb{R})$*, then the action of* (A, h) *on a pair* (τ, z) *in* $\mathfrak{H} \times (L \otimes \mathbb{C})$ *is defined as*

$$((A, h), (\tau, z)) \mapsto (A, h)(\tau, z) := \left(A\tau, \frac{z + x\tau + y}{j(A, \tau)}\right).$$

The real Jacobi group also acts on the space of holomorphic, complex-valued functions defined on $\mathfrak{H} \times (L \otimes \mathbb{C})$.

Definition 1.22 (Jacobi slash operator). Let k be a positive integer and let $\phi : \mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ be a holomorphic function. For every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{R})$, set

$$\phi|_{k,\underline{L}} A(\tau, z) := \phi\left(A\tau, \frac{z}{j(A, \tau)}\right) j(A, \tau)^{-k} e\left(\frac{-c\beta(z)}{j(A, \tau)}\right)$$

and, for every $h = (x, y, \zeta)$ in $H^{\underline{L}}(\mathbb{R})$, set

$$\phi|_{\underline{L}} h(\tau, z) := \zeta \cdot \phi(\tau, z + x\tau + y) \cdot e\left(\tau\beta(x) + \beta(x, z)\right).$$

The action of $J^{\underline{L}}(\mathbb{R})$ on the space of holomorphic, complex-valued functions defined on $\mathfrak{H} \times (L \otimes \mathbb{C})$ is defined as

$$(\phi, (A, h)) \mapsto \phi|_{k,\underline{L}}(A, h) := (\phi|_{k,\underline{L}}A)|_{\underline{L}}h. \tag{1.10}$$

Note that these actions of Γ and $H^{\underline{L}}(\mathbb{R})$ do not commute.

The *integral Jacobi group* is the subgroup $J^{\underline{L}}(\mathbb{Z}) := \mathrm{SL}_2(\mathbb{Z}) \ltimes H^{\underline{L}}(\mathbb{Z})$ of $J^{\underline{L}}(\mathbb{R})$. From now on, drop the word "integral" from the language and the $(\mathbb{Z})$ from the notation for this group.

Definition 1.23 (Jacobi form of lattice index). Let k be a positive integer. A Jacobi form of weight k and index $\underline{L}$ is a holomorphic function $\phi : \mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}$ with the following properties:

(1) for all γ in $J^{\underline{L}}$, the following holds:

$$\phi|_{k,\underline{L}}\gamma(\tau, z) = \phi(\tau, z);$$

(2) the function ϕ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^{\#} \\ n\geq\beta(r)}} c_\phi(n, r)e\left(n\tau + \beta(r, z)\right). \tag{1.11}$$

The complex numbers $c_\phi(\cdot, \cdot)$ are called the *Fourier coefficients* of ϕ.

For fixed weight and index, denote the $\mathbb{C}$-vector space of all such functions by $J_{k,\underline{L}}$.

Remark 1.24. Consider the lattice $\underline{L}_G$ from Example 1.6, (2); then $J_{k,\underline{L}_G}$ is the space $J_{k,\frac{1}{2}G}$ of Jacobi forms of weight k and matrix index $\frac{1}{2}G$ defined in [**BK93**]. Consider the lattice $\underline{L}_m$ from Example 1.6, (1); then the space $J_{k,\underline{L}_m}$ is the space $J_{k,m}$ of Jacobi forms of weight k and scalar index m defined in [**EZ85**].

It is also possible to define Jacobi forms of *half-integral* weight, of *odd* lattice index or with multiplier system. We do not go into further details and instead refer the reader to [**GSZ18**, §III.9]. The following useful result is [**Ajo15**, Proposition 2.4.3]:

Proposition 1.25 ([**Ajo15**, Prop 2.4.3]). *If ϕ in $J_{k,\underline{L}}$ has a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^{\#} \\ n\geq\beta(r)}} c_\phi(n, r)e\left(n\tau + \beta(r, z)\right),$$

then $c_\phi(n, r)$ depends only on $n-\beta(r)$ and on r mod L. More precisely, we have $c_\phi(n, r) = c_\phi(n', r')$ whenever $r \equiv r'$ mod L and $n - \beta(r) = n' - \beta(r')$. Furthermore,

$$c_\phi(n, r) = (-1)^k c_\phi(n, -r).$$

Define the following set, called the support of $\underline{L}$:

$$\mathrm{supp}(\underline{L}) := \{(D, r) : D \in \mathbb{Q}_{\leq 0}, r \in L^{\#}, D \equiv \beta(r) \bmod \mathbb{Z}\}. \tag{1.12}$$

Note that if $(D, r) \in \mathrm{supp}(\underline{L})$, then $D \in \frac{1}{\mathrm{lev}(\underline{L})}\mathbb{Z}$. For every ϕ in $J_{k,\underline{L}}$ with Fourier expansion (1.11) and for each pair (D, r) in $\mathrm{supp}(\underline{L})$, set $C_\phi(D, r) := c_\phi\left(\beta(r) - D, r\right)$. Proposition 1.25 implies that every ϕ in $J_{k,\underline{L}}$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{(D,r)\in\mathrm{supp}(\underline{L})} C_\phi(D, r)e\left((\beta(r) - D)\tau + \beta(r, z)\right). \tag{1.13}$$

We will often use the interplay between these two Fourier expansions. In particular, use the latter to define cusp forms:

Definition 1.26 (Cusp form). A Jacobi form ϕ is called a cusp form if $C_\phi(0,r) = 0$ for all r in $L^\#$ such that $\beta(r) \in \mathbb{Z}$. Denote the $\mathbb{C}$-vector subspace of cusp forms in $J_{k,\underline{L}}$ by $S_{k,\underline{L}}$.

Definition 1.27 (Singular term). For each ϕ in $J_{k,\underline{L}}$, define its singular term as the series

$$C_0(\phi)(\tau,z) := \sum_{\substack{r\in L^\# \\ \beta(r)\in\mathbb{Z}}} C_\phi(0,r)e\left(\tau\beta(r)+\beta(r,z)\right).$$

Definition 1.28. Let r in $L^\#$ be such that $\beta(r) \in \mathbb{Z}$ and define the function

$$g_{\underline{L},r}(\tau,z) := e\left(\tau\beta(r)+\beta(r,z)\right)$$

on the space $\mathfrak{H}\times(L\otimes\mathbb{C})$.

Definition 1.29. Set

$$J_\infty^{\underline{L}} := \left\{\left(\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right),(0,\mu)\right) : n\in\mathbb{Z}, \mu\in L\right\}.$$

We will show in Chapter 2 that $J_\infty^{\underline{L}}$ is the stabilizer of the exponential functions $g_{\underline{L},r}(\cdot,\cdot)$ in $J^{\underline{L}}$. Jacobi–Eisenstein series are defined in the following way:

Definition 1.30 (Jacobi–Eisenstein series). Let k be a positive integer such that $k > \frac{\mathrm{rk}(\underline{L})}{2}+2$. For each r in $\mathrm{Iso}(D_{\underline{L}})$, define the Jacobi–Eisenstein series of weight k and index $\underline{L}$ associated with r as

$$E_{k,\underline{L},r} := \frac{1}{2}\sum_{\gamma\in J_\infty^{\underline{L}}\backslash J^{\underline{L}}} g_{\underline{L},r}|_{k,\underline{L}}\gamma. \tag{1.14}$$

Define the subspace $J_{k,\underline{L}}^{\mathrm{Eis}}$ of $J_{k,\underline{L}}$ as the set $\mathrm{Span}_{\mathbb{C}}\{E_{k,\underline{L},r} : r\in\mathrm{Iso}(D_{\underline{L}})\}$. The series (1.14) converges under the imposed weight restrictions. It is possible to define Jacobi–Eisenstein series for $1\le k\le\frac{\mathrm{rk}(\underline{L})}{2}+2$ by using "Hecke's convergence trick", however we do not pursue this further. It was shown in [**Ajo15**, §3.3] that

$$E_{k,\underline{L},r} = (-1)^k E_{k,\underline{L},-r}. \tag{1.15}$$

Call $E_{k,\underline{L},0}$ the *trivial* Eisenstein series. It is also possible to define "twisted" Eisenstein series:

Definition 1.31. Let k in $\mathbb{N}$ be such that $k>\frac{\mathrm{rk}(\underline{L})}{2}+2$ and let $r\in\mathscr{R}_{\mathrm{Iso}}$. For each primitive Dirichlet character χ modulo F with $F \mid N_r$ and $\chi(-1)=(-1)^k$, define

$$E_{k,\underline{L},r,\chi} := \sum_{d\in\mathbb{Z}_{(N_r)}^\times}\chi(d)E_{k,\underline{L},dr}.$$

For $k\le\mathrm{rk}(\underline{L})+2$, the character χ has to be non-principal (i.e. $F\neq 1$) for convergence reasons.

For every x in $L^\#/L$, define the *Jacobi theta series* associated with x as the function $\vartheta_{\underline{L},x} : \mathfrak{H}\times(L\otimes\mathbb{C})\to\mathbb{C}$,

$$\vartheta_{\underline{L},x}(\tau,z) := \sum_{\substack{r\in L^\# \\ r\equiv x \bmod L}} e\left(\tau\beta(r)+\beta(r,z)\right) \tag{1.16}$$

and set

$$\Theta_{\underline{L}} := \mathrm{Span}_{\mathbb{C}}\{\vartheta_{\underline{L},x} : x\in L^\#/L\}. \tag{1.17}$$

It was shown in [**Boy15**, §3.5] that, for fixed $\underline{L}$, the series $\vartheta_{\underline{L},x}(\tau,\cdot)$ $(x\in L^\#/L)$ are linearly independent as functions of z. These functions are interesting in their own right and much can be said about them. We focus on their modular properties and refer the reader to [**Boy15**, §3 and §4] for an in-depth discussion. Extend the definition of the $|_{k,\underline{L}}$-action of Γ on holomorphic, complex-valued functions defined on $\mathfrak{H} \times (L \otimes \mathbb{C})$ to $\tilde{\Gamma}$ in the following way: for every k in $\frac{1}{2}\mathbb{Z}$ and every $\tilde{A} = (A, w(\tau))$ in $\tilde{\Gamma}$, set

$$\phi|_{k,\underline{L}}\tilde{A}(\tau, z) := \phi\left(A\tau, \frac{z}{w(\tau)^2}\right) w(\tau)^{-2k} e\left(\frac{-c\beta(z)}{w(\tau)^2}\right).$$

It was proved in [**Boy15**, §3.5] that, for every $x \in L^\#/L$ and every $\tilde{A}$ as above, the theta series $\vartheta_{\underline{L},x}$ satisfies the following:

$$\vartheta_{\underline{L},x}|_{\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\tilde{A} = \sum_{y \in L^\#/L} \rho_{\underline{L}}(\tilde{A})_{x,y}\vartheta_{\underline{L},y}. \tag{1.18}$$

In particular, the set $\Theta_{\underline{L}}$ is a $\tilde{\Gamma}$-module. For each ϕ in $J_{k,\underline{L}}$ with Fourier expansion (1.13), define the following function on the upper half-plane:

$$h_{\phi,x}(\tau) = \sum_{\substack{D \in \mathbb{Q} \\ (D,x) \in \mathrm{supp}(\underline{L})}} C_\phi(D, x) q^{-D}.$$

We will review the modular properties of $h_{\phi,x}$ in Subsection 1.3.2. Every Jacobi form has a *theta expansion*:

Proposition ([**Ajo15**, Prop 2.4.7]). *Every Jacobi form ϕ in $J_{k,\underline{L}}$ can be written as*

$$\phi(\tau, z) = \sum_{x \in L^\#/L} h_{\phi,x}(\tau)\vartheta_{\underline{L},x}(\tau, z). \tag{1.19}$$

Theorem 1.32 ([**BS19**, Thm 2.3]). *Let $\underline{L}_1$ and $\underline{L}_2$ be positive-definite, even lattices over $\mathbb{Z}$ and assume that $j : D_{\underline{L}_1} \xrightarrow{\sim} D_{\underline{L}_2}$ is an isomorphism of finite quadratic modules. Then the map*

$$I_j : J_{k+\lceil \frac{rk(\underline{L}_2)}{2} \rceil, \underline{L}_2} \to J_{k+\lceil \frac{rk(\underline{L}_1)}{2} \rceil, \underline{L}_1}$$

defined by

$$\sum_{x \in L_2^\#/L_2} h_{\phi,x}(\tau)\vartheta_{\underline{L}_2,x}(\tau, z) \mapsto \sum_{x \in L_2^\#/L_2} h_{\phi,x}(\tau)\vartheta_{\underline{L}_1,j^{-1}(x)}(\tau, z)$$

is an isomorphism.

Next, define a scalar product on $S_{k,\underline{L}}$. For every τ in $\mathfrak{H}$ and z in $L \otimes \mathbb{C}$, let $\tau = u + iv$ and $z = x + iy$ be their decompositions into real and imaginary parts. In [**Ajo15**, §3.2], the author defines a $J^{\underline{L}}(\mathbb{R})$-invariant *volume element* on $\mathfrak{H} \times (L \otimes \mathbb{C})$ in the following way:

$$dV_{\underline{L},(\tau,z)} := v^{-\mathrm{rk}(\underline{L})-2} du dv dx dy.$$

For every pair of functions ϕ and ψ which are invariant under the $|_{k,\underline{L}}$-action of a subgroup Λ of $J^{\underline{L}}$ of finite index, set

$$\omega_{\phi,\psi}(\tau, z) := \phi(\tau, z)\overline{\psi(\tau, z)} v^k e^{-4\pi\beta(y)v^{-1}}. \tag{1.20}$$

It is easy to check that $\omega_{\phi,\psi}$ is also Λ-invariant.

Definition 1.33 (Petersson scalar product). Let Λ be a subgroup of $J^{\underline{L}}$ of finite index and let $\mathfrak{F}_\Lambda$ denote a fundamental domain for the action of Λ on $\mathfrak{H} \times (L \otimes \mathbb{C})$. If ϕ and ψ are two functions which are invariant under the $|_{k,\underline{L}}$-action of Λ and either one of them is a cusp form, define

$$\langle \phi, \psi \rangle_\Lambda := \frac{1}{[J^{\underline{L}} : \Lambda]} \int_{\mathfrak{F}_\Lambda} \omega_{\phi,\psi}(\tau, z) dV_{\underline{L},(\tau,z)}. \tag{1.21}$$

The Petersson scalar product of two Jacobi forms does not depend on the choice of fundamental domain, or in fact of the subgroup Λ. Thus, drop the subscript from the notation and write $\langle\phi,\psi\rangle := \langle\phi,\psi\rangle_\Lambda$. Given a fundamental domain $\mathfrak{F}$ for the action of Γ on $\mathfrak{H}$ and a fundamental parallelotope $\mathfrak{P}$ for $(L\otimes\mathbb{C})/(\tau L+L)$, choose as a fundamental domain for the action of $J^{\underline{L}}$ on $\mathfrak{H}\times(L\otimes\mathbb{C})$ the set

$$\mathfrak{F}_{J^{\underline{L}}} := \{(\tau,z)\in\mathfrak{H}\times(L\otimes\mathbb{C}) : \tau\in\mathfrak{F}, z\in\mathfrak{P}\}/\{\mathrm{id},\iota\},$$

where ι is the reflection map $(\tau,z)\mapsto(\tau,-z)$. The Petersson scalar product can be expressed in terms of theta expansions in the following way:

Proposition 1.34 ([**Ajo15**, Prop 3.2.10]). *Let*

$$\phi = \sum_{x\in L^\#/L} h_{\phi,x}\vartheta_{\underline{L},x} \qquad \textit{and} \qquad \psi = \sum_{x\in L^\#/L} h_{\psi,x}\vartheta_{\underline{L},x}$$

be Jacobi forms in $J_{k,\underline{L}}$ such that either one of them is a cusp form. Then

$$\langle\phi,\psi\rangle = 2^{-\frac{\mathrm{rk}(\underline{L})}{2}}\det(\underline{L})^{-\frac{1}{2}}\int_{\Gamma\backslash\mathfrak{H}}\sum_{x\in L^\#/L} h_{\phi,x}(\tau)\overline{h_{\psi,x}(\tau)}v^{k-\frac{\mathrm{rk}(\underline{L})}{2}-2}dudv.$$

In the proof of this Proposition given in [**Ajo15**], a scalar product is defined on $\Theta_{\underline{L}}$ by fixing a fibre τ in $\mathfrak{H}$ in (1.21):

$$\langle\sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}, \sum_{s\in L^\#/L} d_s\vartheta_{\underline{L},s}\rangle := \int_{\mathfrak{P}}\sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}(\tau,z)\sum_{s\in L^\#/L}\overline{d_s\vartheta_{\underline{L},s}(\tau,z)}v^{k-\mathrm{rk}(\underline{L})-2}e^{-4\pi\beta(y)v^{-1}}dxdy.$$

It was shown in [**Ajo15**, §3.2] that

$$\langle\sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}, \sum_{s\in L^\#/L} d_s\vartheta_{\underline{L},s}\rangle = v^{k-\frac{\mathrm{rk}(\underline{L})}{2}-2}(2\det(\underline{L}))^{-\frac{\mathrm{rk}(\underline{L})}{2}}\sum_{r\in L^\#/L} c_r\overline{d_r}.$$

Let $[\cdot,\cdot]$ denote the following normalization of the above scalar product on $\Theta_{\underline{L}}$:

$$[\sum_{r\in L^\#/L} c_r\vartheta_{\underline{L},r}, \sum_{s\in L^\#/L} d_s\vartheta_{\underline{L},s}] := \sum_{r\in L^\#/L} c_r\overline{d_r}. \tag{1.22}$$

This scalar product is non-degenerate.

1.3.1. Jacobi forms of scalar index. It is useful to have a background knowledge of the theory of Jacobi forms of scalar index. The integral *scalar Jacobi group* is $\Gamma^J := \Gamma\ltimes\mathbb{Z}^2$. This group acts on the right on the space of holomorphic, complex-valued functions defined on $\mathfrak{H}\times\mathbb{C}$. Let k and m be positive integers. For every $\gamma=(A,h)$ with $A=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ in Γ and $h=(x,y)$ in $\mathbb{Z}^2$, set

$$\phi|_{k,m}\gamma(\tau,z) := \phi\left(A\tau,\frac{z+x\tau+y}{c\tau+d}\right)(c\tau+d)^{-k}e^m\left(\frac{-c(z+x\tau+y)^2}{c\tau+d}+x^2\tau+2xz+xy\right).$$

This action agrees with Definition 1.22 when $\underline{L}=\underline{L}_m$ (see Remark 1.24). The space $J_{k,m}$ of *Jacobi forms of weight k and scalar index m* consists of all holomorphic functions $\phi:\mathfrak{H}\times\mathbb{C}\to\mathbb{C}$ with the following properties:

(1) for all (A,h) in Γ^J, we have $\phi|_{k,m}(A,h)=\phi$;
(2) the function ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{n,r'\in\mathbb{Z}\\ 4mn-r'^2\geq 0}} b_\phi(n,r')e(n\tau+r'z), \qquad \text{where } b_\phi(n,r')\in\mathbb{C}. \tag{1.23}$$

Note that $L_m^\# = \frac{1}{2m}\mathbb{Z}$, $\det(\underline{L}_m) = 2m$ and $\mathrm{lev}(\underline{L}_m) = 4m$. Substitute r' for $2mr$ in (1.11) and set $b_\phi(n, r') := c_\phi(n, \frac{r'}{2m})$, in order to obtain the same expression as above. A scalar Jacobi form is called a *cusp form* if $b_\phi(n, r') = 0$ whenever $4mn = r^2$.

Example 1.35. Let $k \geq 4$ be an even integer. The Fourier expansion of the Eisenstein series $E_{k,\underline{L}_m,0}$ is computed in [**EZ85**, §I.2]:

$$E_{k,\underline{L}_m,0}(\tau, z) = \sum_{\substack{n\in\mathbb{Z}, r\in\frac{1}{2m}\mathbb{Z}\\ n\geq mr^2}} e_{k,m}(n, r)e(\tau n + 2mrz); \tag{1.24}$$

if $n = mr^2$, then $e_{k,m}(n, r) = 1$ if $r \in \mathbb{Z}$ and it is equal to zero otherwise; if $n > mr^2$, then

$$e_{k,m}(n, r) = \frac{(-1)^{\frac{k}{2}}\pi^{k-\frac{1}{2}}}{m^{k-1}2^{k-2}\Gamma(k - \frac{1}{2})}(4nm - 4m^2r^2)^{k-\frac{3}{2}} \sum_{c=1}^{\infty} c^{-k} \sum_{\substack{\lambda, d \bmod c\\ (d,c)=1}} e_c(md^{-1}\lambda^2 - 2mr\lambda + nd),$$

where d^{-1} denotes the inverse of d modulo c. Note that we have made the substitution $r = 2ms$ and relabelled $s = r$ in [**EZ85**, §I.2, (5)]. When $m = 1$, the above expression simplifies to

$$e_{k,1}(n, r) = \frac{L_{4(r^2-n)}(2 - k)}{\zeta(3 - 2k)},$$

where we remind the reader that $L_D(s) := L(s, \chi_D)$ for every discriminant D. When m is square-free,

$$e_{k,m}(n, r) = \frac{1}{\zeta(3 - 2k)\sigma_{k-1}(m)} \sum_{d|(n,2mr,m)} d^{k-1} L_{\frac{4m(mr^2-n)}{d^2}}(2 - k)$$

and it is possible to obtain a similar expression for arbitrary m. We generalize these results in Section 2.3.

In general, write $m = ab^2$, where a is the square-free part of m, and define

$$E_{k,m,s}(\tau, z) := \frac{1}{2} \sum_{\gamma\in J_\infty^{\underline{L}_m}\backslash\Gamma^J} q^{as^2}\zeta^{2abs}|_{k,m}\gamma,$$

where we remind the reader that $q = e(\tau)$ and $\zeta = e(z)$. Then

$$q^{as^2}\zeta^{2abs} = e\left(m\tau\left(\frac{s}{b}\right)^2 + 2mz\frac{s}{b}\right) = g_{\underline{L},\frac{s}{b}}(\tau, z)$$

and the following holds:

$$\left\{\frac{s}{b} : s \in \mathbb{Z}_{(b)}\right\} = \mathrm{Iso}(D_{\underline{L}_m}).$$

To check that this is true, if $\frac{r}{2m} \in L^\#/L$, then $\beta(\frac{r}{2m}) = \frac{r^2}{4m}$ is an integer if and only if $4m \mid r^2$, i.e if and only if $4ab^2 \mid r^2$. This is equivalent to the condition that $r = 2abs$ for some s in $\mathbb{Z}$. It follows that $E_{k,m,s} = E_{k,\underline{L}_m,\frac{s}{b}}$. Twisted scalar Eisenstein series are defined in [**SZ88**, §2] in the following way: for every divisor t of b and every primitive Dirichlet character χ modulo F with $F \mid \frac{b}{t}$ and $\chi(-1) = (-1)^k$, set

$$E_{k,m,t,\chi} := \sum_{d \bmod \frac{b}{t}} \chi(d)E_{k,m,td}.$$

The order of $\frac{t}{b}$ in $L_m^\#/L_m$ is equal to b/t and therefore

$$E_{k,m,t,\chi} = \sum_{d \bmod N_{\frac{t}{b}}} \chi(d)E_{k,\underline{L},\frac{td}{b}}.$$

This does not agree with Definition 1.31, since the coprimality conditions are missing in the summation.

Example 1.36. It is also possible to define scalar Jacobi forms of *half-integral* weight and *half-integral* index. An important example is the scalar Jacobi theta series

$$\vartheta(\tau, z) = \sum_{n\in\mathbb{Z}} \left(\tfrac{-4}{n}\right) e\left(\tau\frac{n^2}{8} + \frac{nz}{2}\right), \tag{1.25}$$

which has weight $\frac{1}{2}$, index $\frac{1}{2}$ and multiplier system

$$v_\vartheta(A, (x, y)) := v_\eta(A)^3 \cdot (-1)^{x+y}.$$

It can be used as a building block for Jacobi forms, together with the Dedekind η-function.

It was proved in [**SZ88**] that there exists a *Hecke equivariant lifting map* between Jacobi forms and elliptic modular forms. Let W_m denote the m-th Atkin–Lehner involution $\left(\begin{smallmatrix} 0 & -1 \\ m & 0 \end{smallmatrix}\right)$ and set

$$M_k^\varepsilon(m) := \mathrm{Span}_\mathbb{C}\{f \in M_k(m) : f|_k W_m = \varepsilon i^{-k} f\},$$

where $\varepsilon \in \{+, -\}$. Then

$$f \in M_k^\varepsilon(m) \implies \Lambda_m(s, f) = \varepsilon\Lambda_m(k - s, f).$$

The space $M_k(m)$ has a (not necessarily unique) basis of modular forms whose L-series have an Euler product. Every such modular form f is an eigenform of all Hecke operators $T(l)$ with $(l, m) = 1$ and has the same eigenvalues for these operators as a unique newform g in $M_k(m')$ for some $m' \mid m$. The quotient $L(f, s)/L(g, s)$ is a finite Dirichlet series with a product expansion of the form

$$\frac{L(f, s)}{L(g, s)} = \prod_{p \mid \frac{m}{m'}} Q_p(s),$$

where $Q_p(s)$ is a polynomial in p^{-s}. The completed L-function of g has a functional equation under $s \mapsto k - s$ and, provided f is an eigenform of W_m, so does $\Lambda_m(f, s)$. When this is the case, each of the Q_p's has a functional equation

$$Q_p(k - s) = \pm p^{-v_p(m'/m)(k-2s)} Q_p(s).$$

Define $\mathfrak{M}_k(m)$ to be the subspace of $M_k(m)$ which is spanned by all f for which the sign in the above equation is $+$ for all $p \mid \frac{m}{m'}$ and set

$$\mathfrak{M}_k^\pm(m) := \mathfrak{M}_k(m) \cap M_k^\pm(m).$$

For the definition of Hecke operators acting on Jacobi forms of lattice index, the reader can consult Definition 3.2. The following holds:

Theorem 1.37 ([**SZ88**, Main Thm (2nd version)]). *For $k \geq 2$, the spaces $J_{k,m}$ and $\mathfrak{M}_{2k-2}^-(m)$ are isomorphic as Hecke modules.*

The lifting map is given below:

Theorem 1.38 ([**SZ88**, Thm 5]). *Let Δ be a fundamental discriminant and let s be an integer such that $\Delta \equiv s^2 \bmod 4m$. Then the map*

$$\mathscr{S}_{\Delta,s} : J_{k,m} \to \mathfrak{M}_{2k-2}^-(m),$$

defined by

$$\sum_{\substack{n,r'\in\mathbb{Z}\\ 4mn-r'^2\geq 0}} b(n,r')q^n\zeta^{r'} \mapsto \sum_{l\geq 0}\left\{\sum_{a|l} a^{k-2}\left(\tfrac{\Delta}{a}\right) b\left(\frac{l^2}{a^2}\cdot\frac{\Delta-s^2}{4m}, \frac{l}{a}s\right)\right\}q^l$$

commutes with Hecke operators and with Atkin–Lehner involutions, it preserves cusp forms and Eisenstein series and a linear combination of these maps is an isomorphism.

Special care needs to be taken when $l = 0$ in the above equation, however we omit the details and refer the reader to [**SZ88**, §3] instead.

1.3.2. Jacobi forms and vector-valued modular forms. In this subsection, we discuss the connection between Jacobi forms and vector-valued modular forms for the dual Weil representation (see Definitions 1.4 and 1.16). We remind the reader of the theta expansion of a Jacobi form ϕ in $J_{k,\underline{L}}$:

$$\phi(\tau,z) = \sum_{x\in L^\#/L} h_{\phi,x}(\tau)\vartheta_{\underline{L},x}(\tau,z).$$

It was proved in [**Boy15**, §3.7] that, for every $x \in L^\#/L$ and every $\tilde{A}$ in $\tilde{\Gamma}$, the functions $h_{\phi,x}$ satisfy the following:

$$h_{\phi,x}|_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\tilde{A} = \sum_{y\in L^\#/L} \overline{\rho_{\underline{L}}(\tilde{A})_{x,y}}h_{\phi,y}.$$

These modular properties imply that the vector-valued function

$$h_{\phi,\underline{L}}(\tau) := \sum_{x\in L^\#/L} h_{\phi,x}(\tau)\mathfrak{e}_x$$

is an element of $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\rho^*_{\underline{L}})$ (see Definition 1.4). Moreover, as a result of (1.18), the vector-valued function

$$\theta_{\underline{L}}(\tau,z) := \sum_{x\in L^\#/L} \vartheta_{\underline{L},x}(\tau,z)\mathfrak{e}_x$$

is an element of $M_{\frac{\mathrm{rk}(\underline{L})}{2}}(\rho_{\underline{L}})$. The main result in [**Boy15**, §3] is the following theorem:

Theorem 1.39 ([**Boy15**, Thm 3.5]). *If $\underline{L} = (L,\beta)$ is a positive-definite, even lattice over $\mathbb{Z}$, then the map*

$$\varphi : \phi \mapsto h_{\phi,\underline{L}}$$

is an isomorphism between $J_{k,\underline{L}}$ and $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\rho^*_{\underline{L}})$.

The results in [**Boy15**] hold over arbitrary totally real number fields, not only over $\mathbb{Q}$. A consequence of this theorem is that $J_{k,\underline{L}} = \{0\}$ if $k < \mathrm{rk}(\underline{L})/2$ and that the spaces $J_{k,\underline{L}}$ are finite-dimensional. When $k \in \mathbb{Z}$, it also gives a connection between Jacobi forms of odd rank lattice index and half-integral weight elliptic modular forms, while for Jacobi forms of even rank lattice index it gives a connection to integral weight elliptic modular forms. For every fixed lattice $\underline{L}$, the value $k = \mathrm{rk}(\underline{L})/2$ is called its *singular weight*. The value $k = (\mathrm{rk}(\underline{L}) + 1)/2$ is called the *critical weight*. Note that there also exists an isomorphism between *skew-holomorphic* Jacobi forms of lattice index $\underline{L}$ and vector-valued modular forms for $\rho_{\underline{L}}$. We do not go into further details and instead refer the reader to [**CS17**, §15.2], for example, where the scalar case is treated.

Another important representation in the theory of vector-valued modular forms is the *Schrödinger representation*. It is typically a representation of the Heisenberg group on the group algebra $\mathbb{C}[D]$ of some finite quadratic module D. Let H be the Heisenberg group $\mathbb{Z}^3$ with the following composition law:

$$(m,n,t)(m',n',t') = (m+m', n+n', t+t'+mn'-nm').$$

Definition 1.40 (Schrödinger representation). Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$ and let $x \in L^\#/L$. The Schrödinger representation of H on $\mathbb{C}[L^\#/L]$ twisted at x is the representation $\sigma_x : H \to \mathrm{Aut}(\mathbb{C}[L^\#/L])$ defined by

$$\sigma_x(m,n,t)\mathfrak{e}_y := e\left(n\beta(x,y) + (t-mn)\beta(x)\right)\mathfrak{e}_{y-mx}.$$

We check that σ_x is indeed a representation: we have $\sigma_x(0,0,0) = I_{\det(\underline{L})}$ and, for arbitrary elements (m,n,t) and (m',n',t') of H, we have

$$\begin{aligned}
\left(\sigma_x(m,n,t)\sigma_x(m',n',t')\right)\mathfrak{e}_y =& \sigma_x(m,n,t)e\left(n'\beta(x,y) + (t'-m'n')\beta(x)\right)\mathfrak{e}_{y-m'x}\\
=& e\left(n'\beta(x,y) + (t'-m'n')\beta(x)\right)\\
& \times e\left(n\beta(x,y-m'x) + (t-mn)\beta(x)\right)\mathfrak{e}_{y-m'x-mx}\\
=& e((n+n')\beta(x,y) + (t+t'+mn'-nm'\\
& -(m+m')(n+n'))\beta(x))\mathfrak{e}_{y-(m+m')x}\\
=& \sigma_x\left((m,n,t)(m',n',t')\right)\mathfrak{e}_y.
\end{aligned}$$

Every element (m,n,t) of H can be written as a product

$$(m,0,0)(0,n,0)(0,0,t).$$

We remind the reader that a representation $\pi : G \to \mathrm{Aut}(V)$ is unitary if and only if $\overline{\pi(g)}^t\pi(g) = I_{\dim(V)}$ for all g in G. Let $\{y_1,\dots,y_{\det(\underline{L})}\}$ denote the elements of $L^\#/L$. Then $\sigma_x(1,0,0)\mathfrak{e}_{y_i} = \mathfrak{e}_{y_i-x}$ and therefore the matrix of $\sigma_x(1,0,0)$ is a permutation matrix (hence it is unitary). Furthermore, $\sigma_x(0,1,0)\mathfrak{e}_{y_i} = e(\beta(x,y_i))\mathfrak{e}_{y_i}$ and $\sigma_x(0,0,1)\mathfrak{e}_{y_i} = e(\beta(x))\mathfrak{e}_{y_i}$, therefore their matrices are diagonal with diagonal entries of modulus equal to one (hence they are unitary). It follows that σ_x is unitary.

Define the following right-action of Γ on H, for every $A = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ in Γ:

$$((m,n,t),A) \mapsto (m,n,t)^A := (ma+nc, mb+nd, t).$$

Lemma ([**Wil18**, Lemma 4]). *For every $\tilde{A}$ in $\tilde{\Gamma}$ and every (m,n,t) in H, the following relation holds between the Weil and the Schrödinger representations:*

$$\rho_{\underline{L}}(\tilde{A})^{-1}\sigma_x(m,n,t)\rho_{\underline{L}}(\tilde{A}) = \sigma_x((m,n,t)^A). \tag{1.26}$$

We include the proof, since it is not given explicitly in [**Wil18**]:

Proof. Check that (1.26) holds for the generators $\tilde{T}$ and $\tilde{S}$ of $\tilde{\Gamma}$:

$$\begin{aligned}
\rho_{\underline{L}}(\tilde{T})^{-1}\sigma_x(m,n,t)\rho_{\underline{L}}(\tilde{T})\mathfrak{e}_y =& \rho_{\underline{L}}(\tilde{T})^{-1}\sigma_x(m,n,t)e(\beta(y))\mathfrak{e}_y\\
=& \rho_{\underline{L}}(\tilde{T})^{-1}e\left(n\beta(x,y) + (t-mn)\beta(x) + \beta(y)\right)\mathfrak{e}_{y-mx}\\
=& e\left(n\beta(x,y) + (t-mn)\beta(x) + \beta(y) - \beta(y-mx)\right)\mathfrak{e}_{y-mx}\\
=& e((m+n)\beta(x,y) + (t-m(m+n))\beta(x))\mathfrak{e}_{y-mx}\\
=& \sigma_x((m,n,t)^T)\mathfrak{e}_y.
\end{aligned}$$

For $\tilde{S}$, it suffices to check that equality holds for the three generators of H. We include the calculations in one of the three cases, since the rest can be treated analogously:

$$\begin{aligned}\rho_{\underline{L}}(\tilde{S})^{-1}\sigma_x(1,0,0)\rho_{\underline{L}}(\tilde{S})\mathfrak{e}_y =&\rho_{\underline{L}}(\tilde{S})^{-1}\sigma_x(1,0,0)\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}}}\sum_{s\in L^\#/L} e(-\beta(y,s))\mathfrak{e}_s\\ =&\rho_{\underline{L}}(\tilde{S})^{-1}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}}}\sum_{s\in L^\#/L} e(-\beta(y,s))\mathfrak{e}_{s-x}\\ =&\frac{1}{\det(\underline{L})}\sum_{s\in L^\#/L}\sum_{r\in L^\#/L} e(\beta(s-x,r)-\beta(y,s))\mathfrak{e}_r\\ =&\frac{1}{\det(\underline{L})}\sum_{s\in L^\#/L} e(\beta(r-y,s))\sum_{r\in L^\#/L} e(-\beta(x,r))\mathfrak{e}_r\\ =&e(-\beta(x,y))\mathfrak{e}_y = \sigma_x((1,0,0)^S)\mathfrak{e}_y,\end{aligned}$$

where we have used the fact that, for every y in $L^\#/L$, we have

$$\sum_{s\in L^\#/L} e(\beta(y,s)) = \begin{cases}\det(\underline{L}), & \text{if } y=0 \text{ and}\\ 0, & \text{otherwise.}\end{cases} \qquad \square$$

Since σ_x is unitary, its dual representation is obtained by complex conjugation:

$$\sigma_x^*(m,n,t)\mathfrak{e}_y = e\left(-n\beta(x,y)+(mn-t)\beta(x)\right)\mathfrak{e}_{y-mx}.$$

Taking complex conjugates on both sides of (1.26), we obtain the following relation between the duals of the Schrödinger and the Weil representations:

$$\overline{\rho_{\underline{L}}(\tilde{A})^{-1}}\sigma_x^*(m,n,t)\rho_{\underline{L}}^*(\tilde{A}) = \sigma_x^*((m,n,t)^A)$$

and therefore

$$\sigma_x^*(m,n,t) = \rho_{\underline{L}}^*(\tilde{A})\sigma_x^*((m,n,t)^A)\rho_{\underline{L}}^*(\tilde{A})^{-1}. \tag{1.27}$$

We will use the Schrödinger representation in Section 2.4 to define an *averaging operator* on $J_{k,\underline{L}}$.

1.3.3. Examples of Jacobi forms. We have seen some examples of Jacobi forms of scalar index in Subsection 1.3.1. We list examples of Jacobi forms for some of the lattices in Example 1.6, as given in [**Gri18**]. We remind the reader of the definitions of the Dedekind η-function (1.6) and the scalar Jacobi theta series (1.25).

EXAMPLE 1.41. For every n in $\mathbb{N}$, τ in $\mathfrak{H}$ and $z=(z_1,\dots,z_n)$ in $\mathbb{C}^n$, define

$$\vartheta_{\mathbb{Z}^n}(\tau,z) := \vartheta(\tau,z_1)\dots\vartheta(\tau,z_n). \tag{1.28}$$

For $1\le n\le 8$, the following function is a Jacobi form of weight $12-n$ and index D_n:

$$\psi_{12-n,D_n}(\tau,z) := \eta(\tau)^{24-3n}\vartheta_{\mathbb{Z}^n}(\tau,z).$$

When $n=8$, this is a Jacobi form of singular weight for D_8. For $n\le 7$, the function ψ_{12-n,D_n} is a cusp form. It is well-known that $D_3=A_3$ and hence we also obtain a cusp form of weight 9 and index A_3.

EXAMPLE 1.42. The function

$$\psi_{4,A_7}(\tau,z) = \vartheta(\tau,z_1)\dots\vartheta(\tau,z_7)\vartheta(\tau,z_1+\dots+z_7)$$

is an element of J_{4,A_7} (we have written $z = (z_1, \dots, z_7)$). Set

$$\Theta_{1,A_2}(\tau, z_1, z_2) := \frac{\vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_1 + z_2)}{\eta(\tau)}.$$

Then $\in J_{1,A_2}(v_\eta^8)$ and

$$\psi_{9,A_2} := \eta^{16}(\tau)\Theta_{1,A_2}(\tau, z_1, z_2)$$

is an element of S_{9,A_2}. We also have that

$$\psi_{6,2A_2}(\tau, z) = \eta^8(\tau)\Theta_{1,A_2}(\tau, z_1, z_2)\Theta_{1,A_2}(\tau, z_3, z_4) \in S_{6,2A_2}$$

(where $nA_2 = \underbrace{A_2 \oplus \cdots \oplus A_2}_{n \text{ times}}$) and that

$$\psi_{3,3A_2}(\tau, z) = \Theta_{1,A_2}(\tau, z_1, z_2)\Theta_{1,A_2}(\tau, z_3, z_4)\Theta_{1,A_2}(\tau, z_5, z_6) \in J_{3,3A_2}.$$

Examples 1.41 and 1.42 are part of the theory of *theta blocks* developed in [**GSZ18**]. Here is another example from [**Gri**]:

Example 1.43. Bearing in mind the modularity properties of theta series (1.18) and the fact that E_8 is a unimodular lattice, the theta series

$$\vartheta_{E_8}(\tau, z) = \sum_{r \in E_8} e\left(\tau\frac{(r, r)}{2} + (r, z)\right) \tag{1.29}$$

is an element of J_{4,E_8}. This is a Jacobi form of singular weight for E_8. Furthermore, fix an element x in E_8 and set $(x, x) = 2m$. Then the following function defined on $\mathfrak{H} \times \mathbb{C}$ is a Jacobi form of weight 4 and scalar index m:

$$\vartheta_{E_8,x}(\tau, z) := \vartheta_{E_8}(\tau, zx).$$

It has a Fourier expansion of the form

$$\vartheta_{E_8,x}(\tau, z) = 1 + \sum_{n \geq 0, l \in \mathbb{Z}} a(n, l)e(n\tau + lz),$$

where

$$a(n, l) = \#\{y \in E_8 : (y, y) = n \text{ and } (x, y) = l\}.$$

Note that the scalar Eisenstein series $E_{4,1,0}$ is equal to $\vartheta_{E_8,(\frac{1}{2},\dots,\frac{1}{2})}$.

In general, if $\underline{L}$ is an even, unimodular lattice, then

$$\vartheta_{\underline{L},0}(\tau, z) = \sum_{r \in L} e(\beta(r)\tau + \beta(r, z))$$

is a Jacobi form of singular weight $\frac{\mathrm{rk}(L)}{2}$ and index $\underline{L}$. The type of construction we encountered in the last example can be extended to arbitrary lattices in the following way: let $\phi \in J_{k,\underline{L}}$ and $\lambda \in L$; for a variable z in $\mathbb{C}$, the function $\phi(\tau, z\lambda)$ is a Jacobi form of weight k and scalar index $\beta(\lambda)$.

CHAPTER 2

Poincaré and Eisenstein series

In this chapter, we define Poincaré series for Jacobi forms of lattice index and show that they reproduce the Fourier coefficients of cusp forms under the Petersson scalar product. We compute the Fourier expansions of Poincaré and Eisenstein series and give an explicit formula for the Fourier coefficients of the trivial Eisenstein series in terms of values of Dirichlet L-functions at negative integers. For even weight and fixed index, we obtain non-trivial linear relations between the Fourier coefficients of non-trivial Eisenstein series and those of the trivial one. This result is used to obtain formulas for the Fourier coefficients of Eisenstein series associated with isotropic elements of small order.

Throughout this chapter, let k be a positive integer and let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$. For every pair (D, r) in the support of $\underline{L}$ (1.12), define the following complex-valued function on the space $\mathfrak{H} \times (L \otimes \mathbb{C})$:

$$g_{\underline{L},D,r}(\tau, z) := e\left(\tau\left(\beta(r) - D\right) + \beta(r, z)\right).$$

Lemma 2.1. *The function $g_{\underline{L},D,r}$ is invariant under the $|_{k,\underline{L}}$-action of $J_\infty^{\underline{L}}$. Furthermore,*

$$g_{\underline{L},D,r}|_{k,\underline{L}}(-I_2)(\tau, z) = (-1)^k g_{\underline{L},D,-r}(\tau, z).$$

Proof. Let $\gamma = \left(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), (0, \mu)\right)$ be an arbitrary element of $J_\infty^{\underline{L}}$. Then

$$g_{\underline{L},D,r}|_{k,\underline{L}}\gamma(\tau, z) = g_{\underline{L},D,r}\left(\tau + n, z + \mu\right) = g_{\underline{L},D,r}(\tau, z),$$

since $e(n(\beta(r) - D)) = e(\beta(r, \mu)) = 1$. For the second part, we have

$$\begin{aligned} g_{\underline{L},D,r}|_{k,\underline{L}}(-I_2)(\tau, z) =& (-1)^{-k} e(\tau(\beta(r) - D) + \beta(r, -z)) \\ =& (-1)^{-k} e(\tau(\beta(-r) - D) + \beta(-r, z)) = (-1)^k g_{\underline{L},D,-r}(\tau, z). \end{aligned}$$ □

Note that the function $g_{\underline{L},r}(\cdot, \cdot)$ from Definition 1.28 is equal to $g_{\underline{L},0,r}(\cdot, \cdot)$.

2.1. Poincaré series

In this section, we define Jacobi–Poincaré series of lattice index and deduce some of their properties, using the methods in [**BK93**].

Definition 2.2. Let the pair (D, r) in $\operatorname{supp}(\underline{L})$ be such that $D < 0$. Define the Jacobi–Poincaré series of weight k and index $\underline{L}$ associated with the pair (D, r) as the series

$$P_{k,\underline{L},D,r}(\tau, z) := \sum_{\gamma \in J_\infty^{\underline{L}} \backslash J^{\underline{L}}} g_{\underline{L},D,r}|_{k,\underline{L}}\gamma(\tau, z). \tag{2.1}$$

As a consequence of Lemma 2.1, the series (2.1) is independent of the choice of coset representatives of $J_\infty^{\underline{L}} \setminus J^{\underline{L}}$. The same lemma also implies that

$$P_{k,\underline{L},D,-r} = (-1)^k P_{k,\underline{L},D,r}, \tag{2.2}$$

as was the case for Eisenstein series. The following theorem is the main result of this section:

THEOREM 2.3. *Let k be a positive integer and let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$. The Poincaré series $P_{k,\underline{L},D,r}$ satisfies the following:*

(i) If $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$, then $P_{k,\underline{L},D,r}$ is absolutely and uniformly convergent on compact subsets of $\mathfrak{H} \times (L \otimes \mathbb{C})$ and it is an element of $S_{k,\underline{L}}$. Furthermore, define the constant

$$\lambda_{k,\underline{L},D} := 2^{-2k+\frac{\mathrm{rk}(\underline{L})}{2}+2}\Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2} - 1\right)\det(\underline{L})^{-\frac{1}{2}}(\pi|D|)^{-k+\frac{\mathrm{rk}(\underline{L})}{2}+1}.$$

For every cusp form ϕ in $S_{k,\underline{L}}$ with Fourier expansion

$$\phi(\tau, z) = \sum_{(D',r') \in \mathrm{supp}(\underline{L})} C_\phi(D', r') e\left((\beta(r') - D')\tau + \beta(r', z)\right), \tag{2.3}$$

the following holds:

$$\langle \phi, P_{k,\underline{L},D,r} \rangle = \lambda_{k,\underline{L},D} C_\phi(D, r).$$

(ii) For every (D, r) and (D', r') in $\mathrm{supp}(\underline{L})$ such that $D < 0$, set

$$\delta_{\underline{L}}(D, r, D', r') := \begin{cases} 1, & \text{if } D' = D \text{ and } r' \equiv r \bmod L \text{ and} \\ 0, & \text{otherwise} \end{cases} \tag{2.4}$$

and

$$\begin{aligned} G_{k,\underline{L},D,r}(D', r') :=& \delta_{\underline{L}}(D, r, D', r') + (-1)^k \delta_{\underline{L}}(D, -r, D', r') + \frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}} \\ & \times \left(\frac{D'}{D}\right)^{\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} - \frac{1}{2}} \sum_{c \geq 1} c^{-\frac{\mathrm{rk}(\underline{L})}{2} - 1} J_{k - \frac{\mathrm{rk}(\underline{L})}{2} - 1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right) \\ & \times \left(H_{\underline{L},c}(D, r, D', r') + (-1)^k H_{\underline{L},c}(D, -r, D', r')\right), \end{aligned} \tag{2.5}$$

where the function J_α is the J-Bessel function of index α defined in (1.1) *and*

$$\begin{aligned} H_{\underline{L},c}(D, r, D', r') := \sum_{d \in \mathbb{Z}_{(c)}^\times, \lambda \in L/cL} & e_c\big((\beta(\lambda + r) - D)d^{-1} \\ & + (\beta(r') - D')d + \beta(r', \lambda + r)\big). \end{aligned} \tag{2.6}$$

The Poincaré series $P_{k,\underline{L},D,r}$ has the following Fourier expansion:

$$P_{k,\underline{L},D,r}(\tau, z) = \sum_{\substack{(D',r') \in supp(\underline{L}) \\ D' < 0}} G_{k,\underline{L},D,r}(D', r') e\left((\beta(r') - D')\tau + \beta(r', z)\right).$$

Note that (2.2) follows from (ii). Furthermore, it is clear from the definitions of $\delta_L(D, r, D', r')$ and $H_{\underline{L},c}(D, r, D', r')$ that $P_{k,\underline{L},D,r}$ only depends on $r \bmod L$. As a consequence of this fact and of (i), we obtain the following corollary:

COROLLARY 2.4. *The set*

$$\left\{P_{k,\underline{L},D,r} : r \in L^\# / L, D \in \mathbb{Q}_{<0}, \beta(r) \equiv D \bmod \mathbb{Z}\right\}$$

generates $S_{k,\underline{L}}$.

PROOF. (i) Choose

$$\left\{\left(A, (\lambda, 0)^A\right) : A \in \mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z}), \lambda \in L\right\}$$

as a set of coset representatives for $J^{\underline{L}}_{\infty}\backslash J^{\underline{L}}$ and note that $(\lambda, 0)^A = (a\lambda, b\lambda)$ for every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{Z})$. The series (2.1) can be written as

$$P_{k,\underline{L},D,r}(\tau, z) = \sum_{\lambda\in L, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} e\left(\frac{-c\beta(z + (a\tau + b)\lambda)}{c\tau + d} + a^2\tau\beta(\lambda) + a\beta(\lambda, z)\right)$$
$$\times e\left((\beta(r) - D)\frac{a\tau + b}{c\tau + d} + \beta\left(r, \frac{z + (a\tau + b)\lambda}{c\tau + d}\right)\right)(c\tau + d)^{-k}.$$

Using the fact that

$$a^2\tau(c\tau + d) - c(a\tau + b)^2 = a\tau - abc\tau - b(ad - 1) = (a\tau + b) - ab(c\tau + d)$$

and the following well-known identity for the modular action of Γ on $\mathfrak{H}$:

$$\frac{a\tau + b}{c\tau + d} = \frac{a}{c} - \frac{1}{c(c\tau + d)}, \tag{2.7}$$

we obtain that

$$\begin{aligned} P_{k,\underline{L},D,r}(\tau, z) &= \sum_{\lambda\in L, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} (c\tau + d)^{-k} e\Bigg(\beta(z)\frac{-c}{c\tau + d} + \beta(\lambda)\frac{a\tau + b}{c\tau + d} + \frac{\beta(\lambda, z)}{c\tau + d} \\ &\qquad + (\beta(r) - D)\frac{a\tau + b}{c\tau + d} + \frac{\beta(r, z)}{c\tau + d} + \beta(r, \lambda)\frac{a\tau + b}{c\tau + d}\Bigg) \\ &= \sum_{\lambda\in L, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma} e\left(-D\frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^{-k} e\left(\frac{-c\beta(z)}{c\tau + d}\right) \\ &\qquad \times e\left(\beta(\lambda + r)\frac{a\tau + b}{c\tau + d} + \frac{\beta(\lambda + r, z)}{c\tau + d}\right) \\ &= \sum_{A\in\Gamma_\infty\backslash\Gamma} e\left(-DA\tau\right) j(A, \tau)^{-k} e\left(\frac{-cz}{j(A, \tau)}\right)\vartheta_{\underline{L},r}(A(\tau, z)), \end{aligned} \tag{2.8}$$

after rearranging terms and where c denotes the bottom-left entry of A in the above equation. For every $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ, let $\tilde{A}$ denote the canonical lifting $(\sqrt{c\tau + d}, A)$ to $\tilde{\Gamma}$. Using (1.18), we obtain that

$$\begin{aligned} P_{k,\underline{L},D,r}(\tau, z) &= \sum_{A\in\Gamma_\infty\backslash\Gamma} e\left(-DA\tau\right) j(A, \tau)^{-\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)}\vartheta_{\underline{L},r}|_{\frac{\mathrm{rk}(\underline{L})}{2},\underline{L}}\tilde{A}(\tau, z) \\ &= \sum_{A\in\Gamma_\infty\backslash\Gamma} e\left(-DA\tau\right) j(A, \tau)^{-\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} \sum_{y\in L^\#/L} \rho_{\underline{L}}(\tilde{A})_{r,y}\vartheta_{\underline{L},y}(\tau, z). \end{aligned} \tag{2.9}$$

The image of $\rho_{\underline{L}}$ is finite, by [**Boy15**, Theorem 3.4, (*ii*)]. It follows that the inner sum in the above equation is bounded by above on compact subsets of $\mathfrak{H} \times L \otimes \mathbb{C}$ by a constant which is independent of A (but dependant on the subset). Coset representatives of $\mathrm{SL}_2(\mathbb{Z})_\infty\backslash \mathrm{SL}_2(\mathbb{Z})$ are well-known and given by matrices $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $(c, d) = 1$ and, for each pair (c, d), choose a and b in $\mathbb{Z}$ such that $ad - bc = 1$. It is also well-known that, for every lattice Λ in $\mathbb{C}$, the series

$$\sum_{\omega\in\Lambda\backslash\{0\}} |\omega|^{-s}$$

converges absolutely for $\Re(s) > 2$ (see [**CS17**, Lemma 2.1.6], for example). Combining this with the fact that

$$|e(-DA\tau)| = \exp\left(2\pi D\Im(A\tau)\right) \le 1$$

for every D in $\mathbb{Q}_{\leq 0}$, we obtain that $P_{k,\underline{L},D,r}$ converges uniformly and absolutely on compact subsets of $\mathfrak{H} \times L \otimes \mathbb{C}$ for $k - \frac{\mathrm{rk}(\underline{L})}{2} > 2$.

Provided we show that each $P_{k,\underline{L},D,r}$ is invariant under the $|_{k,\underline{L}}$-action of $J^{\underline{L}}$, the fact that it is an element of $S_{k,\underline{L}}$ follows from inspecting its Fourier expansion (*ii*). For every δ in $J^{\underline{L}}$, we have

$$\begin{aligned} P_{k,\underline{L},D,r}|_{k,\underline{L}}\delta(\tau,z) &= \sum_{\gamma \in J^{\underline{L}}_\infty \backslash J^{\underline{L}}} g_{\underline{L},D,r}|_{k,\underline{L}}\gamma|_{k,\underline{L}}\delta(\tau,z) = \sum_{\gamma \in J^{\underline{L}}_\infty \backslash J^{\underline{L}}} g_{\underline{L},D,r}|_{k,\underline{L}}(\gamma\delta)(\tau,z) \\ &= \sum_{\gamma' \in J^{\underline{L}}_\infty \backslash J^{\underline{L}}} g_{\underline{L},D,r}|_{k,\underline{L}}\gamma'(\tau,z) = P_{k,\underline{L},D,r}(\tau,z), \end{aligned}$$

since right multiplication by δ is an automorphism of $J^{\underline{L}}_\infty \backslash J^{\underline{L}}$.

Let ϕ in $S_{k,\underline{L}}$ have Fourier expansion (2.3). Insert the definition of $P_{k,\underline{L},D,r}$ into the definition of the Petersson scalar product of ϕ and $P_{k,\underline{L},D,r}$ given in (1.21) and interchange the order of integration and summation, using the fact that the integrand converges absolutely and uniformly:

$$(2.10) \quad \langle \phi, P_{k,\underline{L},D,r} \rangle = \int_{\mathfrak{F}_{J^{\underline{L}}}} \omega_{\phi,P_{k,\underline{L},D,r}}(\tau,z) dV_{\underline{L},(\tau,z)} = \sum_{\gamma \in J^{\underline{L}}_\infty \backslash J^{\underline{L}}} \int_{\mathfrak{F}_{J^{\underline{L}}}} \omega_{\phi, g_{\underline{L},D,r}|_{k,\underline{L}}\gamma}(\tau,z) dV_{\underline{L},(\tau,z)}$$

We claim that

$$\omega_{\phi,g_{D,r}|_{k,\underline{L}}\gamma}(\tau,z) = \omega_{\phi,g_{D,r}}(\gamma(\tau,z))$$

for every γ in $J^{\underline{L}}$. Suppose that $\gamma = (A,(\lambda,\mu))$, with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in Γ and (λ,μ) in $H^{\underline{L}}(\mathbb{Z})$. Definition (1.20) implies that

$$\begin{aligned} \omega_{\phi,g_{D,r}|_{k,\underline{L}}\gamma}(\tau,z) =& \phi(\tau,z)\overline{(c\tau+d)^{-k}} \overline{e\left(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,z)\right)} \\ & \times \overline{g_{\underline{L},D,r}(\gamma(\tau,z))} v^k e^{-4\pi\beta(y)v^{-1}} \\ =& \phi(\tau,z) \frac{(c\tau+d)^k}{|c\tau+d|^{2k}} \frac{\left| e\left(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,z)\right)\right|^2}{e\left(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,z)\right)} \\ & \times \overline{g_{\underline{L},D,r}(\gamma(\tau,z))} v^k e^{-4\pi\beta(y)v^{-1}}. \end{aligned}$$

Since ϕ is invariant under the $|_{k,\underline{L}}$-action of $J^{\underline{L}}$, $\Im(A\tau) = \frac{\Im(\tau)}{|c\tau+d|^2}$ for every τ in $\mathfrak{H}$ and every A in Γ and $|e(z)|^2 = e^{-4\pi\Im(z)}$ for every z in $\mathbb{C}$, it follows that

$$\begin{aligned} \omega_{\phi,g_{D,r}|_{k,\underline{L}}\gamma}(\tau,z) =& \phi(\gamma(\tau,z))\overline{g_{\underline{L},D,r}(\gamma(\tau,z))}\Im(A\tau)^k \\ & \times \exp\left(-4\pi\left(\beta(y)v^{-1} + \Im\left(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,z)\right)\right)\right). \end{aligned}$$

For z_1, z_2 in $\mathbb{C}$, λ in L and z in $L \otimes \mathbb{C}$, the following equalities hold:

$$\begin{aligned} &\Im(z_1 z_2) = \Re(z_1)\Im(z_2) + \Im(z_1)\Re(z_2), \\ &\Im(\beta(\lambda,z)) = \beta(\lambda,\Im(z)), \\ &\Re(\beta(z)) = \beta(\Re(z)) - \beta(\Im(z)) \text{ and} \\ &\Im(\beta(z)) = \beta(\Re(z),\Im(z)). \end{aligned}$$

Thus,

$$\begin{aligned}
&\beta(y)v^{-1} + \Im\left(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,z)\right) \\
&= \frac{1}{v|c\tau+d|^2}\Big(|c\tau+d|^2\beta(y+v\lambda) + c^2v^2\beta\left(\Re(z+\lambda\tau+\mu)\right) - c^2v^2 \\
&\quad \times \beta\left(\Im(z+\lambda\tau+\mu)\right) - cv(cu+d)\beta\left((\Re(z+\lambda\tau+\mu), \Im(z+\lambda\tau+\mu)\right)\Big) \\
&= \beta\left(\Im\left(\frac{z+\lambda\tau+\mu}{c\tau+d}\right)\right)\Im(A\tau)^{-1}.
\end{aligned}$$

Substitute (τ, z) for $\gamma(\tau, z)$ in (2.10). Since the volume element $V_{\underline{L},(\tau,z)}$ is invariant under this change of variable, we obtain by the usual unfolding argument that

$$\begin{aligned}
\langle \phi, P_{k,\underline{L},D,r}\rangle &= \sum_{\gamma\in J^{\underline{L}}_\infty\backslash J^{\underline{L}}} \int_{\gamma\mathfrak{F}_{J^{\underline{L}}}} \phi(\tau,z)\overline{g_{\underline{L},D,r}(\tau,z)}v^k e^{-4\pi\beta(y)v^{-1}}\, dV_{\underline{L},(\tau,z)} \\
&= \int_{\mathfrak{F}_{J^{\underline{L}}_\infty}} \phi(\tau,z)\overline{g_{\underline{L},D,r}(\tau,z)}v^k e^{-4\pi\beta(y)v^{-1}}\, dV_{\underline{L},(\tau,z)},
\end{aligned}$$

where $\mathfrak{F}_{J^{\underline{L}}_\infty}$ denotes a fundamental domain for the action of $J^{\underline{L}}_\infty$ on $\mathfrak{H}\times(L\otimes\mathbb{C})$. Each $\gamma = \left(\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right), (0,\mu)\right)$ in $J^{\underline{L}}_\infty$ acts on $\mathfrak{H}\times(L\otimes\mathbb{C})$ via

$$(\gamma, (\tau, z)) \mapsto (\tau+n, z+\mu)\,.$$

Therefore, choose

$$\mathfrak{F}_{J^{\underline{L}}_\infty} = \{(\tau,z)\in\mathfrak{H}\times(L\otimes\mathbb{C}) : 0\le u\le 1, v>0, 0\le x_i\le 1, y\in L\otimes\mathbb{R}\},$$

where we remind the reader that we write $\tau = u+iv$ and $z = x+iy$. It is straight-forward to check that every pair (τ', z') in $\mathfrak{H}\times(L\otimes\mathbb{C})$ can be written as $\gamma(\tau,z)$ for some γ in $J^{\underline{L}}_\infty$ and some unique (τ, z) in our chosen fundamental domain. Insert the Fourier expansion of ϕ to obtain that

$$\begin{aligned}
\langle \phi, P_{k,\underline{L},D,r}\rangle &= \int_0^1\int_0^\infty\int_{[0,1]^{\mathrm{rk}(\underline{L})}}\int_{\mathbb{R}^{\mathrm{rk}(\underline{L})}} \sum_{(D',r')\in\mathrm{supp}(\underline{L})} C_\phi(D',r')e((\beta(r')-D')\tau \\
&\quad + \beta(r',z))\overline{e\left((\beta(r)-D)\tau+\beta(r,z)\right)}v^{k-\mathrm{rk}(\underline{L})-2}e^{-4\pi\beta(y)v^{-1}}\,dydxdvdu \\
&= \sum_{(D',r')\in\mathrm{supp}(\underline{L})} C_\phi(D',r')\int_0^1\int_0^\infty\int_{[0,1]^{\mathrm{rk}(\underline{L})}}\int_{\mathbb{R}^{\mathrm{rk}(\underline{L})}} \\
&\quad \times e\left(u\left[(\beta(r')-D')-(\beta(r)-D)\right]\right)e^{-2\pi v((\beta(r')-D')-(\beta(r)-D))} \\
&\quad \times e(\beta(r'-r,x)+i\beta(r+r',y))v^{k-\mathrm{rk}(\underline{L})-2}e^{-4\pi\beta(y)v^{-1}}\,dydxdvdu \\
&= C_\phi(D,r)\int_0^\infty e^{-4\pi(\beta(r)-D)v}v^{k-\mathrm{rk}(\underline{L})-2}\int_{\mathbb{R}^{\mathrm{rk}(\underline{L})}} e^{-4\pi(\beta(r,y)+\beta(y)v^{-1})}\,dydv.
\end{aligned}$$

In order to obtain the last formula, we have used the orthogonality relations for the complex exponential function

$$\int_0^1 e(u(n'-n))du = \begin{cases}1, & \text{if } n=n' \text{ and}\\ 0, & \text{otherwise}\end{cases} \tag{2.11}$$

and the fact that a similar result holds in higher dimensions:

$$\int_{[0,1]^{\mathrm{rk}(\underline{L})}} e(\beta(r'-r,x))dx = \begin{cases}1, & \text{if } r\equiv r' \bmod L \text{ and}\\ 0, & \text{otherwise.}\end{cases} \tag{2.12}$$

One way to prove (2.12) is by writing the Gram matrix of β in *Smith normal form*, i.e. write $G = U\mathfrak{D}V$, such that $U = (u_{ij})_{i,j}$ and $V = (v_{ij})_{i,j}$ are matrices in $\mathrm{GL}_{\mathrm{rk}(\underline{L})}(\mathbb{Z})$ and $\mathfrak{D} = \mathrm{diag}(d_1, \ldots, d_{\mathrm{rk}(\underline{L})})$, where $d_1 \mid d_2 \mid \cdots \mid d_{\mathrm{rk}(\underline{L})}$ are the *elementary divisors* of G. It is clear that (2.11) implies that

$$\int_{[0,1]^{\mathrm{rk}(\underline{L})}} e(\beta(L, x))dx = 1.$$

Every $s \neq 0$ in $L^{\#}/L$ can be written as $s = d_k\lambda$, where d_k is one of the elementary divisors of G and $\lambda \in L$. It follows that

$$\begin{aligned}\int_{[0,1]^{\mathrm{rk}(\underline{L})}} e(\beta(s, x))dx &= \int_{[0,1]^{\mathrm{rk}(\underline{L})}} e\left([U^t s]^t \mathfrak{D}(Vx)\right) dx \\ &= \det(V) \prod_{i=1}^{\mathrm{rk}(\underline{L})} \int_{V_{1i}}^{V_{2i}} e\left(d_i[U^t s]_i x_i'\right) dx_i' \\ &= \det(V) \prod_{i=1}^{\mathrm{rk}(\underline{L})} \frac{1}{2\pi i d_i [U^t s]_i} e\left(x_i'\right) \Big|_{d_i[U^t s]_i V_{1i}}^{d_i[U^t s]_i V_{2i}},\end{aligned}$$

under the substitution $x' = Vx$ and where $[V_{1i}, V_{2i}] = (v_{1i}, \ldots, v_{\mathrm{rk}(\underline{L})i})([0, 1]^{\mathrm{rk}(\underline{L})})^t$. In particular, V_{1i} and V_{2i} are integers for all i in $\{1, \ldots, \mathrm{rk}(\underline{L})\}$. When $i = k$, we have $d_i[U^t s]_i \in \mathbb{Z}$ and therefore the corresponding term in the above product vanishes.

Since $\underline{L}$ is positive-define, its Gram matrix G can be diagonalized with a real orthogonal matrix, i.e. $G = Q^t \mathscr{D} Q$ for some $Q = (q_{ij})_{i,j}$ in $\mathrm{M}_{\mathrm{rk}(\underline{L})}(\mathbb{R})$ which satisfies $Q^t Q = I_{\mathrm{rk}(\underline{L})}$ and some diagonal matrix $\mathscr{D} = \mathrm{diag}(\alpha_1, \ldots, \alpha_{\mathrm{rk}(\underline{L})})$. It follows that $\prod_{j=1}^{\mathrm{rk}(\underline{L})} \alpha_j = \det(\underline{L})$. Thus,

$$\begin{aligned}I :&= \int_{\mathbb{R}^{\mathrm{rk}(\underline{L})}} e^{-4\pi(\beta(r,y)+\beta(y)v^{-1})} dy = \int_{\mathbb{R}^{\mathrm{rk}(\underline{L})}} e^{-4\pi(r^t(Q^t\mathscr{D}Q)y + \frac{1}{2}v^{-1}y^t(Q^t\mathscr{D}Q)y)} dy \\ &= \int_{\mathbb{R}^{\mathrm{rk}(\underline{L})}} e^{-4\pi((Qr)^t\mathscr{D}y' + \frac{1}{2}v^{-1}y'^t\mathscr{D}y')} dy,\end{aligned}$$

under the substitution $y' = Qy$. Writing the exponents explicitly as functions of the individual vector components and dropping the primes yields

$$I = \int_{\mathbb{R}^{\mathrm{rk}(\underline{L})}} e^{-4\pi\left(\sum_{j=1}^{\mathrm{rk}(\underline{L})} \alpha_j (Qr)_j y_j + (2v)^{-1} \sum_{j=1}^{\mathrm{rk}(\underline{L})} \alpha_j y_j^2\right)} dy = \prod_{j=1}^{\mathrm{rk}(\underline{L})} \left(\int_{\mathbb{R}} e^{-2\pi\alpha_j(2(Qr)_j y_j + v^{-1}y_j^2)} dy_j \right).$$

Complete the square in the exponent and obtain that

$$I = \prod_{j=1}^{\mathrm{rk}(\underline{L})} \left(e^{2\pi\alpha_j v (Qr)_j^2} \int_{\mathbb{R}} e^{-2\pi\alpha_j v^{-1}(y_j + v(Qr)_j)^2} dy_j \right) = \prod_{j=1}^{\mathrm{rk}(\underline{L})} \left(e^{2\pi\alpha_j v (Qr)_j^2} \int_{\mathbb{R}} e^{-2\pi\alpha_j v^{-1} y_j^2} dy_j \right),$$

by a simple change of variable. Substitute x_j for $(2\pi\alpha_j v^{-1})^{\frac{1}{2}} y_j$ and use the standard Gaussian integral to obtain that

$$I = e^{4\pi v \frac{1}{2} \sum_{j=1}^{\mathrm{rk}(\underline{L})} \alpha_j (Qr)_j^2} \prod_{j=1}^{\mathrm{rk}(\underline{L})} \left(\frac{v}{2\alpha_j} \right)^{\frac{1}{2}} = 2^{-\frac{\mathrm{rk}(\underline{L})}{2}} \det(\underline{L})^{-\frac{1}{2}} v^{\frac{\mathrm{rk}(\underline{L})}{2}} e^{4\pi v \beta(r)}.$$

Thus,

$$\langle \phi, P_{k,\underline{L},D,r} \rangle = C_\phi(D, r) 2^{-\frac{\mathrm{rk}(\underline{L})}{2}} \det(\underline{L})^{-\frac{1}{2}} \int_0^\infty e^{-4\pi v D} v^{k - \frac{\mathrm{rk}(\underline{L})}{2} - 2} dv = C_\phi(D, r) \lambda_{k,\underline{L},D},$$

as claimed. Thus, Jacobi–Poincaré series reproduce the Fourier coefficients of Jacobi cusp forms under the Petersson scalar product up to a constant depending on the weight, the index and the Poincaré series itself (note however that it does not depend on r).

(*ii*) Insert the standard choice of coset representatives of $\mathrm{SL}_2(\mathbb{Z})_\infty \backslash \mathrm{SL}_2(\mathbb{Z})$ into (2.8), i.e.

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : (c,d) = 1 \text{ and } a, b \text{ are chosen such that } ad - bc = 1 \right\},$$

and split this sum up into two sums, according to whether $c = 0$ or $c \neq 0$.

When $c = 0$, we have $d = \pm 1$, $a = d = \pm 1$ and b can be any integer. Since $r \in L^\#$ and $\lambda \in L$, we have that $\beta(r,\lambda), \beta(\lambda) \in \mathbb{Z}$ and we obtain the following contribution:

$$\begin{aligned} &\sum_{\lambda \in L} \Big[e\left(\beta(\lambda)\tau + \beta(\lambda, z) + (\beta(r) - D)\tau + \beta(r,z) + \beta(r,\lambda)\tau\right) \\ &\quad + (-1)^{-k} e(\beta(\lambda)\tau - \beta(\lambda, z) + (\beta(r) - D)\tau - \beta(r,z) + \beta(r,\lambda)\tau) \Big] \\ &= \sum_{\lambda \in L} e\left((\beta(\lambda + r) - D)\tau\right) \Big[e(\beta(\lambda + r, z)) + (-1)^k e(\beta(-(\lambda + r), z)) \Big]. \end{aligned}$$

In order to express this as a standard Fourier expansion of a Jacobi form, set $r' := \lambda + r$, which implies that we are summing over all r' in $L^\#$ such that $r' \equiv r \bmod L$. Then introduce an additional summation over all D' in $\mathbb{Q}$ with $D' < 0$ such that $(D', r') \in \operatorname{supp}(\underline{L})$ and impose the condition that $D' = D$. The contribution becomes

$$\sum_{\substack{(D',r') \in \operatorname{supp}(\underline{L}) \\ D' < 0}} e\left((\beta(r') - D')\tau + \beta(r', z)\right) \Big[\delta_{\underline{L}}(D, r, D', r') + (-1)^k \delta_{\underline{L}}(D, r, D', -r') \Big],$$

where δ_L is defined in (2.4).

For the contribution coming from terms with $c \neq 0$, Lemma 2.1 implies that the terms with $c < 0$ are obtained from those with $c > 0$, by multiplying their contribution with $(-1)^k$ and replacing z with $-z$. Thus, concentrate on the former case. Set $n := \beta(r) - D$ and use (2.7) to write the contribution coming from terms with $c > 0$ as

$$\begin{aligned} &\sum_{\substack{c > 0 \\ (c,d) = 1}} \sum_{\lambda \in L} (c\tau + d)^{-k} e\bigg(\frac{-c}{c\tau + d} \beta\left(z - \frac{1}{c}\lambda\right) + \frac{a}{c}\beta(\lambda) \\ &\quad + \beta\left(r, \frac{1}{c\tau + d}\left(z - \frac{1}{c}\lambda\right) + \frac{a}{c}\lambda\right) + n\left(\frac{a}{c} - \frac{1}{c(c\tau + d)}\right)\bigg). \end{aligned}$$

Write d as $d' + \alpha c$, where d' is the reduction of d modulo c and $\alpha \in \mathbb{Z}$. As d runs through $\mathbb{Z}$ with the condition that $(d, c) = 1$ in the above equation, the new variable d' runs through congruence classes modulo c which are coprime to c (drop the prime from the notation for simplicity) and α runs through $\mathbb{Z}$. Similarly, write λ as $\lambda' + \mu c$, where λ' is the reduction of λ modulo cL and $\mu \in L$. The new variable λ' runs through the coset representatives of L/cL (drop the prime from the notation) and μ runs through L. We obtain the contribution

$$\begin{aligned}
&\sum_{\substack{c>0,\alpha\in\mathbb{Z},d\in\mathbb{Z}_{(c)}^{\times}\\ \mu\in L,\lambda\in L/cL}} c^{-k}\left(\tau+\frac{d}{c}+\alpha\right)^{-k} e\Bigg(\frac{-1}{\tau+\frac{d}{c}+\alpha}\beta\left(z-\frac{1}{c}\lambda-\mu\right)+\frac{a}{c}\beta(\lambda)\\
&\quad+\frac{1}{c(\tau+\frac{d}{c}+\alpha)}\beta\left(r,z-\frac{1}{c}\lambda-\mu\right)+\frac{a}{c}\beta(r,\lambda)+n\left(\frac{a}{c}-\frac{1}{c^2(\tau+\frac{d}{c}+\alpha)}\right)\Bigg)\\
&=\sum_{c>0}c^{-k}\sum_{d\in\mathbb{Z}_{(c)}^{\times},\lambda\in L/cL} e_c\left(\left(\beta(\lambda)+\beta(r,\lambda)+n\right)d^{-1}\right)F_{k,\underline{L},c;(n,r)}\left(\tau+\frac{d}{c},z-\frac{\lambda}{c}\right),
\end{aligned}$$

where d^{-1} denotes the inverse of d modulo c. Furthermore, the function $F_{k,\underline{L},c;(n,r)}$: $\mathfrak{H}\times(L\otimes\mathbb{C})\to\mathbb{C}$ is defined as

$$F_{k,\underline{L},c;(n,r)}(\tau,z):=\sum_{\alpha\in\mathbb{Z},\mu\in L}(\tau+\alpha)^{-k}e\left(\frac{-1}{\tau+\alpha}\beta(z-\mu)+\frac{\beta(r,z-\mu)}{c(\tau+\alpha)}-\frac{n}{c^2(\tau+\alpha)}\right).$$

It has period $\mathbb{Z}$ in τ and period L in z and hence it has a Fourier expansion of the form

$$F_{k,\underline{L},c;(n,r)}(\tau,z)=\sum_{n'\in\mathbb{Z},r'\in L^{\#}}f(n',r')e\left(n'\tau+\beta(r',z)\right).$$

Since $P_{k,\underline{L},D,r}$ is absolutely and uniformly convergent in τ and z on compact subsets of $\mathfrak{H}\times(L\otimes\mathbb{C})$, so is $F_{k,\underline{L},c;(n,r)}$. Hence, its Fourier coefficients can be computed by integrating it against an appropriate exponential function. For every fixed $v>0$, y in $L\otimes\mathbb{R}$, m in $\mathbb{Z}$ and s in $L^{\#}$, we have

$$\begin{aligned}
&\int_{[0,1]}\int_{[0,1]^{\mathrm{rk}(\underline{L})}}F_{k,\underline{L},c;(n,r)}(u+iv,x+iy)e\left(-mu-\beta(s,x)\right)dxdu\\
&\quad=\sum_{n'\in\mathbb{Z},r'\in L^{\#}}f(n',r')e\left(in'v+\beta(r',iy)\right)\int_{[0,1]}e\left((n'-m)u\right)du\\
&\quad\times\int_{[0,1]^{\mathrm{rk}(\underline{L})}}e\left(\beta(r'-s,x)\right)dx=f(m,s)e(imv)e\left(\beta(s,iy)\right),
\end{aligned}$$

using (2.11) and (2.12). Thus, evaluate $f(n',r')$ as

$$\begin{aligned}
f(n',r')&=\sum_{\alpha\in\mathbb{Z},\mu\in L}\int_{[0,1]}\int_{[0,1]^{\mathrm{rk}(\underline{L})}}(\tau+\alpha)^{-k}e\Bigg(\frac{-1}{\tau+\alpha}\beta(z-\mu)+\frac{1}{c(\tau+\alpha)}\beta(r,z-\mu)\\
&\quad-\frac{n}{c^2(\tau+\alpha)}\Bigg)e(-n'\tau)e\left(-\beta(r',z)\right)dxdu\\
&=\int_{-\infty}^{\infty}\tau^{-k}e(-n'\tau)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e\left(\frac{-1}{\tau}\beta(z)+\frac{1}{c\tau}\beta(r,z)-\frac{n}{c^2\tau}-\beta(r',z)\right)dxdu,
\end{aligned}$$

where we write $\tau=u+iv$ and $z=x+iy$ as usual. Make the change of variable $z\mapsto z+\frac{1}{c}r-\tau r'$. Expanding the integrand, the inner multiple integral is equal to

$$\begin{aligned}
&\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e\left(\frac{-1}{\tau}\beta(z)+\frac{1}{c^2\tau}\beta(r)+\tau\beta(r')-\frac{1}{c}\beta(r,r')-\frac{n}{c^2\tau}\right)dx\\
&=e_c\left(-\beta(r',r)\right)e\left(\tau\beta(r')+\frac{D}{c^2\tau}\right)\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}e\left(\frac{-1}{\tau}\beta(z)\right)dx.
\end{aligned}$$

The multiple integral in x is standard and can be computed by diagonalizing β and using the generalized Gaussian integral

$$\int_{-\infty}^{\infty} \exp\left(\frac{1}{2}iax^2 + iJx\right)dx = \left(\frac{2\pi i}{a}\right)^{\frac{1}{2}} \exp\left(\frac{-iJ^2}{2a}\right), \qquad a, J \in \mathbb{C}.$$

We have

$$\begin{aligned}
\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e\left(\frac{-1}{\tau}\beta(z)\right)dx &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} e\left(\frac{-z^t(Q^t\mathscr{D}Q)z}{\tau}\right)dx \\
&= e^{\frac{\pi i}{\tau}y^tGy}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\frac{-\pi i}{\tau}(Qx)^t\mathscr{D}(Qx) + \frac{2\pi}{\tau}(Qx)^t\mathscr{D}(Qy)\right)dx \\
&= e^{\frac{\pi i}{\tau}y^tGy}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\sum_{j=i}^{\mathrm{rk}(\underline{L})}\left(\frac{-\pi i}{\tau}\alpha_j x_j^2 + \frac{2\pi}{\tau}\alpha_j(Qy)_j x_j\right)\right)dx \\
&= e^{\frac{\pi i}{\tau}y^tGy}\prod_{j=i}^{\mathrm{rk}(\underline{L})}\int_{-\infty}^{\infty} \exp\left(\frac{-\pi i}{\tau}\alpha_j x_j^2 + \frac{2\pi}{\tau}\alpha_j(Qy)_j x_j\right)dx_j \\
&= e^{\frac{\pi i}{\tau}y^tGy}\prod_{j=i}^{\mathrm{rk}(\underline{L})}\left(\frac{2\pi i}{\frac{-2\pi\alpha_j}{\tau}}\right)^{\frac{1}{2}} \exp\left(\frac{-i\left(\frac{2\pi\alpha_j}{i\tau}(Qy)_j\right)^2}{2\cdot\frac{-2\pi\alpha_j}{\tau}}\right) \\
&= e^{\frac{\pi i}{\tau}y^tGy}\exp\left(\frac{-\pi i}{\tau}\sum_{j=1}^{\mathrm{rk}(\underline{L})}\alpha_j(Qy)_j^2\right)\prod_{j=1}^{\mathrm{rk}(\underline{L})}\left(\frac{\tau}{i\alpha_j}\right)^{\frac{1}{2}} = \det(\underline{L})^{-\frac{1}{2}}\left(\frac{\tau}{i}\right)^{\frac{\mathrm{rk}(\underline{L})}{2}},
\end{aligned}$$

since $\prod_j \alpha_j = \det(\underline{L})$ and $\sum_j \alpha_j(Qy)_j^2 = \beta(y,y)$. Set $D' := \beta(r') - n'$. Thus,

$$f(n', r') = \det(\underline{L})^{-\frac{1}{2}} e_c(-\beta(r', r)) \int_{-\infty}^{\infty}\left(\frac{\tau}{i}\right)^{\frac{\mathrm{rk}(\underline{L})}{2}} \tau^{-k} e\left(D'\tau + \frac{D}{c^2\tau}\right)du.$$

Consider the cases $D' \geq 0$ and $D' < 0$ separately. We remind the reader that $v = \Im(\tau)$ is fixed.

If $D' \geq 0$, then let $R > 0$ and consider the closed contour integral

$$\oint \frac{1}{(u+iv)^{k-\frac{\mathrm{rk}(\underline{L})}{2}}} e\left(D'(u+iv) + \frac{D}{c^2(u+iv)}\right)du, \tag{2.13}$$

over the contour C from Figure 1, formed by traversing the line segment

$$L = \{\sigma : -R \leq \sigma \leq R\}$$

from left to right and the semi-circle

$$S = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$$

in the counter-clockwise direction.

The integrand is holomorphic inside C and therefore (2.13) is equal to zero by Cauchy's Theorem. The integral we seek is

$$\lim_{R\to\infty}\int_L \frac{1}{(u+iv)^{k-\frac{\mathrm{rk}(\underline{L})}{2}}} e\left(D'(u+iv) + \frac{D}{c^2(u+iv)}\right)du \tag{2.14}$$

and hence we need to estimate the integral over S as $R \to \infty$. The estimation lemma from complex analysis implies that its absolute value is less than or equal to

$$\pi R \max_{u\in S}\left|\frac{1}{(u+iv)^{k-\frac{\mathrm{rk}(\underline{L})}{2}}} e\left(D'(u+iv) + \frac{D}{c^2(u+iv)}\right)\right|.$$

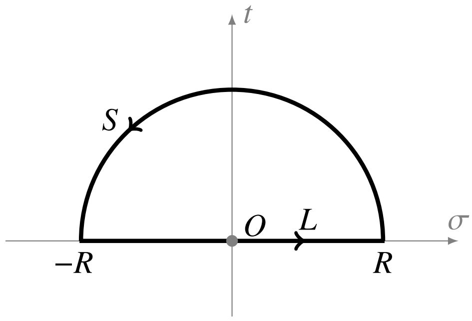

FIGURE 1. The contour C

With the chosen parametrization of $u = Re^{i\theta}$ $(0 \le \theta \le \pi)$, we have

$$\frac{R}{|u+iv|^{k-\frac{\mathrm{rk}(\underline{L})}{2}}} = \frac{R}{|R\cos\theta + i(R\sin\theta + v)|^{k-\frac{\mathrm{rk}(\underline{L})}{2}}} = \frac{R}{(R^2 + 2Rv\sin\theta + v^2)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}}}.$$

For fixed R and v, the denominator reaches its minimum when $\sin\theta$ reaches its minimum value of zero. Since $k > \mathrm{rk}(\underline{L}) + 2$ by assumption, we have

$$\lim_{R\to\infty} \frac{R}{(R^2+v^2)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}}} = 0.$$

Moving on to the exponential term, we have:

$$\begin{aligned}\left|e\left(D'(u+iv)\right)\right| &= \left|\exp\left(2\pi i D' R(\cos\theta + i\sin\theta) - 2\pi D' v\right)\right| \\ &= \left|\exp\left(-2\pi D' R\sin\theta\right)\right| \left|\exp(-2\pi D' v)\right|,\end{aligned}$$

and the maximum of this expression is equal to $\exp(-2\pi D' v)$, since $D' \ge 0$ and $0 \le \sin\theta \le 1$ for $0 \le \theta \le \pi$. For the final estimate, we have:

$$\begin{aligned}\left|e\left(\frac{D}{c^2(u+iv)}\right)\right| &= \left|\exp\left(\frac{2\pi i D}{c^2(R\cos\theta + i(R\sin\theta+v))}\right)\right| \\ &= \left|\exp\left(\frac{2\pi i D R\cos\theta}{c^2(R^2\cos^2\theta + (R\sin\theta+v)^2)} + \frac{2\pi D(R\sin\theta+v)}{c^2(R^2\cos^2\theta + (R\sin\theta+v)^2)}\right)\right| \\ &= \left|\exp\left(\frac{2\pi D(R\sin\theta+v)}{c^2(R^2+2Rv\sin\theta+v^2)}\right)\right|,\end{aligned}$$

and the maximum of this expression converges to 1 as $R \to \infty$, by a similar argument as above. Thus, the integral over S converges to zero as $R \to \infty$. Therefore, so does (2.14) and hence $f(n', r')$ vanishes when $D' \ge 0$.

If $D' < 0$, then substitute $\frac{i}{c}\left(\frac{D}{D'}\right)^{1/2} s$ for τ and write $s = \sigma + it$. Integrating in u from $-\infty$ to ∞ implies that we are integrating in t from ∞ to $-\infty$ and we obtain that

$$\begin{aligned}f(n', r') &= \det(\underline{L})^{-\frac{1}{2}} e_c\left(-\beta(r', r)\right) i^{-k} c^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \left(\frac{D}{D'}\right)^{\frac{\mathrm{rk}(\underline{L})}{4}-\frac{k}{2}+\frac{1}{2}} (-1) \int_{\infty}^{-\infty} s^{\frac{\mathrm{rk}(\underline{L})}{2}-k} \\ &\quad \times e\left(-\frac{(-D')i}{c}\left(\frac{D}{D'}\right)^{1/2} s + \frac{(-D)}{c^2}\left[\frac{i}{c}\left(\frac{D}{D'}\right)^{1/2} s\right]^{-1}\right) dt \\ &= \det(\underline{L})^{-\frac{1}{2}} e_c\left(-\beta(r', r)\right) i^{-k} c^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}} \\ &\quad \times \int_{-\infty}^{\infty} s^{-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)} \exp\left(\frac{2\pi(DD')^{\frac{1}{2}}}{c}\left(s - s^{-1}\right)\right) dt.\end{aligned}$$

For fixed $\omega > 0$ and $\kappa > 0$, the functions

$$f(t) = \left(\frac{t}{\kappa}\right)^{\frac{\omega-1}{2}} J_{\omega-1}(2\sqrt{\kappa t}),\ t > 0, \qquad \text{and} \qquad F(s) = s^{-\omega} e^{-\frac{\kappa}{s}},\ \Re(s) > 0,$$

are mutually inverse with respect to the Laplace transform, i.e.

$$f(t) = \frac{1}{2\pi i}\int_{T-i\infty}^{T+i\infty} F(s)e^{st}ds, \qquad T > 0.$$

Taking $t = \kappa = \frac{2\pi(DD')^{1/2}}{c}$ and $\omega = k - \frac{\mathrm{rk}(\underline{L})}{2}$, it follows that

$$f(n', r') = \frac{2\pi i^{-k}}{\det(\underline{L})^{\frac{1}{2}}} e_c\left(-\beta(r', r)\right) c^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}} J_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right)$$

(note that $ds = idt$) and the terms with $c > 0$ give the following contribution:

$$\sum_{c>0}\sum_{d\in\mathbb{Z}^\times_{(c)},\lambda\in L/cL} e_c\Big((\beta(\lambda)+\beta(r,\lambda)+n)\,d^{-1} - \beta(r', r)\Big) \sum_{\substack{n'\in\mathbb{Z}, r'\in L^\# \\ \beta(r')<n'}} \frac{2\pi i^{-k}}{\det(\underline{L})^{\frac{1}{2}}} c^{-\frac{\mathrm{rk}(\underline{L})}{2}-1}$$
$$\times \left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}} J_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right) e\left(n'\left(\tau+\frac{d}{c}\right)+\beta\left(r', z-\frac{\lambda}{c}\right)\right).$$

Writing $i^{-1} = -i$ and substituting $\lambda \mapsto -\lambda$ in the lattice sum, the above is equal to

$$\sum_{(D',r')\in\mathrm{supp}(\underline{L})} \frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}} \sum_{c>0} c^{-\frac{\mathrm{rk}(\underline{L})}{2}-1} J_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right)(-1)^k$$
$$\times \sum_{d\in\mathbb{Z}^\times_{(c)},\lambda\in L/cL} e_c\Big((\beta(\lambda-r)-D)d^{-1} + (\beta(r')-D')d + \beta(r', \lambda-r)\Big)$$
$$\times\, e\left((\beta(r')-D')\tau + \beta(r', z)\right).$$

Terms with $c < 0$ give the same contribution, multiplied by $(-1)^k$ and with z replaced by $(-z)$ and therefore they give a contribution of

$$\sum_{(D',r')\in\mathrm{supp}(\underline{L})} \frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}} \sum_{c>0} c^{-\frac{\mathrm{rk}(\underline{L})}{2}-1} J_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right)$$
$$\times \sum_{d\in\mathbb{Z}^\times_{(c)},\lambda\in L/cL} e_c\Big((\beta(\lambda-r)-D)d^{-1} + (\beta(r')-D')d + \beta(r', \lambda-r)\Big)$$
$$\times\, e\left((\beta(r')-D')\tau + \beta(r', -z)\right)$$
$$= \sum_{(D',r')\in\mathrm{supp}(\underline{L})} \frac{2\pi (-i)^k}{\det(\underline{L})^{\frac{1}{2}}}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}} \sum_{c>0} c^{-\frac{\mathrm{rk}(\underline{L})}{2}-1} J_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right)$$
$$\times \sum_{d\in\mathbb{Z}^\times_{(c)},\mu\in L/cL} e_c\Big((\beta(\mu+r)-D)d^{-1} + (\beta(-r')-D')d + \beta(-r', \mu+r)\Big)$$
$$\times\, e\left((\beta(-r')-D')\tau + \beta(-r', z)\right).$$

We have made the substitution $\mu = -\lambda$. Substitute r' for $-r'$ in the above by abuse of notation. The observation that $H_{\underline{L},c}(D, r, D', -r') = H_{\underline{L},c}(D, -r, D', r')$ for every r and r' in $L^\#$ concludes the proof. ◻

The lattice sum $H_{\underline{L},c}(D,r,D',r')$ can be re-written in terms of Kloosterman sums (1.2) as

$$H_{\underline{L},c}(D,r,D',r') = \sum_{\lambda\in L/cL} e_c(\beta(r',\lambda+r))S(\beta(r')-D',\beta(\lambda+r)-D;c).$$

As a consequence of Theorem 2.3, the following estimates for the Fourier coefficients of Jacobi cusp forms hold:

Proposition 2.5. *Suppose that $k > \mathrm{rk}(\underline{L})+2$. Let ϕ in $S_{k,\underline{L}}$ have Fourier expansion* (2.3)*. Then there exists an $\epsilon > 0$ such that*

$$C_\phi(D',r') \ll_{\epsilon,k} \left(1 + 2^{\frac{\mathrm{rk}(\underline{L})^2}{2}+\mathrm{rk}(\underline{L})+\epsilon}\det(\underline{L})^{\epsilon-\frac{1}{2}}|D'|^{\frac{\mathrm{rk}(\underline{L})}{2}+\epsilon}\right)^{1/2}$$
$$\times \det(\underline{L})^{\frac{1}{4}}2^{\frac{k}{2}-\mathrm{rk}(\underline{L})-\frac{1}{2}+k\frac{\mathrm{rk}(\underline{L})}{2}-\frac{\mathrm{rk}(\underline{L})^2}{4}}|D'|^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{1}{2}}\sqrt{\langle\phi,\phi\rangle}.$$

This result is [**BK93**, §1, Proposition 1], under the substitutions $G = 2m$, $r' = G^{-1}r^t$ and $D' = \frac{-D}{2\det(\underline{L})}$.

It is also possible to define *Jacobi–Poincaré square series*, which were studied in [**Wil18**] for vector-valued modular forms for the Weil representation and used to construct *automorphic products*. For every r in $L^\#/L$ and every D in $\mathbb{Q}_{<0}$ such that $\beta(r) \equiv D \bmod \mathbb{Z}$, define the Poincaré square series of weight k and index $\underline{L}$ associated with the pair (D,r) as

$$Q_{k,\underline{L},D,r} := \sum_{n\in\mathbb{Z}} P_{k,\underline{L},n^2D,nr},$$

where $P_{k,\underline{L},0,0} := E_{k,\underline{L},0}$. This series converges absolutely for $k > \frac{\mathrm{rk}(\underline{L})}{2}+3$ and, in view of (2.2), it can be written as

$$Q_{k,\underline{L},D,r} = E_{k,\underline{L},0} + \sum_{n\in\mathbb{N}}\left(1+(-1)^k\right)P_{k,\underline{L},n^2D,nr}.$$

When k is odd, equation (1.15) implies that $E_{k,\underline{L},0}$ vanishes identically and hence so does $Q_{k,\underline{L},D,r}$. When k is even, we obtain that

$$Q_{k,\underline{L},D,r} = E_{k,\underline{L},0} + 2\sum_{n\in\mathbb{N}} P_{k,\underline{L},n^2D,nr}$$

and, using (1.4), that

$$\frac{1}{2}\sum_{d\in\mathbb{N}}\mu(d)\left(Q_{k,\underline{L},d^2D,dr} - E_{k,\underline{L},0}\right) = \sum_{d\in\mathbb{N}}\mu(d)\sum_{n\in\mathbb{N}}P_{k,\underline{L},d^2n^2D,dnr} = \sum_{n\in\mathbb{N}}\sum_{d|n}\mu(d)P_{k,\underline{L},n^2D,nr}$$
$$= P_{k,\underline{L},D,r}.$$

In other words, Poincaré series can be recovered from the Poincaré square series.

2.2. Eisenstein series

In this section, we prove analogous results to Theorem 2.3 for Jacobi–Eisenstein series:

Theorem 2.6. *Let k be a positive integer and let $\underline{L} = (L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$. The Eisenstein series $E_{k,\underline{L},r}$ satisfies the following:*

(i) If $k > \frac{\mathrm{rk}(\underline{L})}{2}+2$, then $E_{k,\underline{L},r}$ is absolutely and uniformly convergent on compact subsets of $\mathfrak{H}\times(L\otimes\mathbb{C})$ and it is an element of $J_{k,\underline{L}}$. Furthermore, it is orthogonal to cusp forms. That is, for every ϕ in $S_{k,\underline{L}}$, the following holds:

$$\langle\phi,E_{k,\underline{L},r}\rangle = 0.$$

(ii) For every (D', r') *in* $\mathrm{supp}(\underline{L})$*, set*

$$G_{k,\underline{L},r}(D',r') := \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{2\det(\underline{L})^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)} \times \sum_{c\geq 1} c^{-k}\left(H_{\underline{L},c}(r,D',r') + (-1)^k H_{\underline{L},c}(-r,D',r')\right), \tag{2.15}$$

where

$$H_{\underline{L},c}(r,D',r') := \sum_{d\in\mathbb{Z}^\times_{(c)},\lambda\in L/cL} e_c\left(\beta(\lambda+r)d^{-1} + (\beta(r')-D')d + \beta(r',\lambda+r)\right). \tag{2.16}$$

The Eisenstein series $E_{k,\underline{L},r}$ *has the following Fourier expansion:*

$$E_{k,\underline{L},r}(\tau,z) = \frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z) + (-1)^k\vartheta_{\underline{L},-r}(\tau,z)\right) + \sum_{\substack{(D',r')\in supp(\underline{L})\\ D'<0}} G_{k,\underline{L},r}(D',r')e\left((\beta(r')-D')\tau + \beta(r',z)\right),$$

where $\vartheta_{\underline{L},r}$ *is a theta series as in* (1.16).

Note that $H_{\underline{L},c}(r,D',r') = H_{\underline{L},c}(0,r,D',r')$ with the notation in (2.6) and that (1.15) follows from (*ii*). Furthermore, it is clear from the definition of $H_{\underline{L},c}(r,D',r')$ that $E_{k,\underline{L},r}$ only depends on r mod L (this fact was also pointed out in [**Ajo15**]).

Proof. (*i*) The same arguments used in the proof of Theorem 2.3, (*i*), imply that the series defined in (1.14) converges absolutely and uniformly on compact subsets of $\mathfrak{H}\times(L\otimes\mathbb{C})$ for $k > \frac{\mathrm{rk}(\underline{L})}{2}+2$. It was also shown that it is independent of the choice of coset representatives of $J^{\underline{L}}_\infty \setminus J^{\underline{L}}$ and that it is invariant under the $|_{k,\underline{L}}$-action of $J_{k,\underline{L}}$. The fact that it is an element of $J_{k,\underline{L}}$ follows from inspecting its Fourier expansion (*ii*).

We remind the reader that the Petersson scalar product of two Jacobi forms converges if either one of them is a cusp form. The Petersson scalar product of $E_{k,\underline{L},r}$ and an arbitrary cusp form ϕ in $S_{k,\underline{L}}$ can be computed in the same way as in the proof of Theorem 2.3, (*i*). Since ϕ is a cusp form, its Fourier coefficients $C_\phi(0,r)$ vanish and therefore so does $\langle\phi, E_{k,\underline{L},r}\rangle$.

(*ii*) This can be proved by following the steps in the proof of Theorem 2.3, (*ii*), up to a certain point. We pick up from where the differences arise. When analysing the contribution coming from terms with $c = 0$ in the Fourier expansion of $E_{k,\underline{L},r}$, set $r' := \lambda + r$ as before, which implies we are summing over r' in $L^\#$ such that $r' \equiv r \bmod L$. Since $D = 0$ in this case, the contribution is equal to

$$\frac{1}{2}\sum_{\substack{r'\in L^\#\\ r'\equiv r \bmod L}} e\left(\beta(r')\tau\right)\left(e\left(\beta(r',z)\right) + (-1)^k e\left(\beta(-r',z)\right)\right) = \frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau,z) + (-1)^k\vartheta_{\underline{L},-r}(\tau,z)\right),$$

as claimed. In the contribution coming from terms with $c \neq 0$, the change arises in the Fourier coefficients of $F_{k,\underline{L},c;(n,r)}$. They are now given by the equation

$$f(n',r') = \det(\underline{L})^{-\frac{1}{2}} e_c\left(-\beta(r',r)\right)\int_{-\infty}^{\infty}\left(\frac{\tau}{i}\right)^{\frac{\mathrm{rk}(\underline{L})}{2}} \tau^{-k} e\left(D'\tau\right) du,$$

where $\tau = u + iv$ and $v > 0$ is fixed. If $D' \geq 0$, then applying similar estimates to before yields $f(n', r') = 0$. When $D' < 0$, we need to compute the integral

$$I := \int_{-\infty+iv}^{\infty+iv} \left(\frac{\tau}{i}\right)^{\frac{\mathrm{rk}(\underline{L})}{2}} \tau^{-k} e\left(D'\tau\right) d\tau.$$

Substitute s for $2\pi i D'\tau$ in order to obtain that

$$I = i^{1-k}(-2\pi D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \int_{C+i\infty}^{C-i\infty} s^{-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)} e^s ds,$$

where $C := -2\pi D' v$ is a positive constant. For fixed $\nu > 0$, the functions

$$f(t) = t^{\nu-1} \qquad \text{and} \qquad F(s) = \Gamma(\nu)s^{-\nu}$$

are mutually inverse with respect to the Laplace transform. Taking $\nu = k - \frac{\mathrm{rk}(\underline{L})}{2}$ implies that

$$I = i^{1-k}(-2\pi D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \frac{(-2\pi i) f(1)}{\Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} = \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^{-k}}{\Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}$$

and hence

$$f(n', r') = \det(\underline{L})^{-\frac{1}{2}} e_c\left(-\beta(r', r)\right) \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^{-k}}{\Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}.$$

Thus, terms with $c > 0$ give the following contribution:

$$\begin{aligned}
&\sum_{c>0} c^{-k} \sum_{d \in \mathbb{Z}_{(c)}^{\times}, \lambda \in L/cL} e_c\left((\beta(\lambda) + \beta(r, \lambda) + \beta(r)) d^{-1} - \beta(r', r)\right) \sum_{\substack{n' \in \mathbb{Z}, r' \in L^{\#} \\ \beta(r') < n'}} i^{-k} \\
&\times \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}} \left(k - \frac{\mathrm{rk}(\underline{L})}{2} - 1\right)!} e\left(n'\left(\tau + \frac{d}{c}\right) + \beta\left(r', z - \frac{\lambda}{c}\right)\right) \\
&= \sum_{(D', r') \in \mathrm{supp}(\underline{L})} \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}} \left(k - \frac{\mathrm{rk}(\underline{L})}{2} - 1\right)!} \sum_{c>0} c^{-k}(-1)^k \sum_{d \in \mathbb{Z}_{(c)}^{\times}, \lambda \in L/cL} \\
&\times e_c\left(\beta(\lambda - r)d^{-1} + (\beta(r') - D')d + \beta(r', \lambda - r)\right) e\left((\beta(r') - D')\tau + \beta(r', z)\right).
\end{aligned}$$

Terms with $c < 0$ give the same contribution, multiplied by $(-1)^k$ and with z replaced by $(-z)$ and therefore they give a contribution of

$$\begin{aligned}
&\sum_{(D', r') \in \mathrm{supp}(\underline{L})} \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} \sum_{c>0} c^{-k} \sum_{d \in \mathbb{Z}_{(c)}^{\times}, \lambda \in L/cL} \\
&\times e_c\left(\beta(\lambda - r)d^{-1} + (\beta(r') - D')d + \beta(r', \lambda - r)\right) e\left((\beta(r') - D')\tau + \beta(r', -z)\right) \\
&= \sum_{(D', r') \in \mathrm{supp}(\underline{L})} \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D')^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} \sum_{c>0} c^{-k} \sum_{d \in \mathbb{Z}_{(c)}^{\times}, \mu \in L/cL} e_c\left(\beta(\mu + r)d^{-1}\right) \\
&\times e_c\left((\beta(-r') - D')d + \beta(-r', \mu + r)\right) e\left((\beta(-r') - D')\tau + \beta(-r', z)\right).
\end{aligned}$$

We have made the substitution $\mu = -\lambda$. Substitute r' for $-r'$ in the above by abuse of notation, in order to obtain the desired result and complete the proof. $\square$

REMARK 2.7. For $k > \operatorname{rk}(\underline{L}) + 2$, the non-singular Fourier coefficients of $E_{k,\underline{L},r}$ can be obtained from those of $P_{k,\underline{L},D,r}$. Consider (D, r) and (D', r') in $\operatorname{supp}(\underline{L})$ such that $D, D' < 0$ and let $c \in \mathbb{N}$. We have

$$\frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}}\left(\frac{D'}{D}\right)^{\frac{k}{2}-\frac{\operatorname{rk}(\underline{L})}{4}-\frac{1}{2}} c^{-\frac{\operatorname{rk}(\underline{L})}{2}-1} J_{k-\frac{\operatorname{rk}(\underline{L})}{2}-1}\left(\frac{4\pi(DD')^{\frac{1}{2}}}{c}\right)$$

$$= \frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}}\left(\frac{-D'}{-D}\right)^{\frac{k}{2}-\frac{\operatorname{rk}(\underline{L})}{4}-\frac{1}{2}} c^{-\frac{\operatorname{rk}(\underline{L})}{2}-1} \sum_{n\geq 0} \frac{(-1)^n}{n!\Gamma\left(n+k-\frac{\operatorname{rk}(\underline{L})}{2}\right)}\left(\frac{2\pi(-D)^{\frac{1}{2}}(-D')^{\frac{1}{2}}}{c}\right)^{2n+k-\frac{\operatorname{rk}(\underline{L})}{2}-1}$$

$$= \frac{i^k}{\det(\underline{L})^{\frac{1}{2}}} \sum_{n\geq 0} \frac{(-1)^n(2\pi)^{2n+k-\frac{\operatorname{rk}(\underline{L})}{2}}(-D')^{n+k-\frac{\operatorname{rk}(\underline{L})}{2}-1}(-D)^n c^{-2n-k}}{n!\Gamma\left(n+k-\frac{\operatorname{rk}(\underline{L})}{2}\right)}.$$

The J-Bessel function J_α is finite at the point $x = 0$ for positive α. Hence, view D as a parameter in $\mathbb{R}$ and take the limit as $D \to 0$ in (2.5), using the above calculation:

$$\lim_{D\to 0} G_{k,\underline{L},D,r}(D', r') = \frac{i^k(2\pi)^{k-\frac{\operatorname{rk}(\underline{L})}{2}}(-D')^{k-\frac{\operatorname{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}}\Gamma\left(k-\frac{\operatorname{rk}(\underline{L})}{2}\right)} \sum_{c\geq 1} c^{-k} H_{\underline{L},c}(0, r, D', r').$$

The singular term of $E_{k,\underline{L},r}$ as given in Theorem 2.6 is

$$C_0(E_{k,\underline{L},r})(\tau, z) = \frac{1}{2}\left(\vartheta_{\underline{L},r}(\tau, z) + (-1)^k\vartheta_{\underline{L},-r}(\tau, z)\right). \tag{2.17}$$

The following result was stated in the proof of [**Ajo15**, Lemma 3.3.14]:

LEMMA 2.8. *The Eisenstein series $E_{k,\underline{L},r}$ are linearly independent for r in the set* $\operatorname{Iso}(D_{\underline{L}})/\{\pm 1\}$. *Furthermore, if k is even, then $J^{\mathrm{Eis}}_{k,\underline{L}}$ is spanned by*

$$\left\{E_{k,\underline{L},r} : r \in \operatorname{Iso}(D_{\underline{L}})/\{\pm 1\}\right\}$$

and if k is odd, then $J^{\mathrm{Eis}}_{k,\underline{L}}$ is spanned by

$$\left\{E_{k,\underline{L},r} : r \in \operatorname{Iso}(D_{\underline{L}})/\{\pm 1\}, r \not\equiv -r \bmod L\right\}.$$

We include the proof:

PROOF. Suppose that k is even. Equation (1.15) implies that $E_{k,\underline{L},r} = E_{k,\underline{L},-r}$ and therefore all Eisenstein series are represented by the set

$$\left\{E_{k,\underline{L},s} : s \in \operatorname{Iso}(D_{\underline{L}})/\{\pm 1\}\right\}.$$

The elements of this set are linearly independent, as a consequence of (2.17) and of the fact that theta series are linearly independent (see [**Boy15**, §3.5]).

If k is odd, then (1.15) implies that $E_{k,\underline{L},r} = -E_{k,\underline{L}-r}$ and therefore $E_{k,\underline{L},r} = 0$ if $r \equiv -r \bmod L$. Hence, all Eisenstein series are represented by the set

$$\left\{E_{k,\underline{L},s} : s \in \operatorname{Iso}(D_{\underline{L}})/\{\pm 1\} : s \not\equiv -s \bmod L\right\}.$$

Linear independence is a consequence of the fact that theta series are linearly independent. □

We would like to obtain a closed formula for (2.15). The first observation regarding the non-singular Fourier coefficients of $E_{k,\underline{L},r}$ is that (2.16) can be re-written in terms of the Kloosterman sums (1.2) in the following way:

$$H_{\underline{L},c}(r, D', r') = \sum_{\lambda(c)} e_c(\beta(r', \lambda + r))S(\beta(r') - D', \beta(\lambda + r); c)$$

One way to simplify (2.15) is inspired by calculations for scalar Jacobi–Eisenstein series from [**EZ85**, §I.2]. Let b be a positive integer. For every quadratic polynomial $Q : L \to \mathbb{Z}$, define the *representation numbers*

$$R_b(Q) := \#\{\lambda \in L/bL : Q(\lambda) \equiv 0 \bmod b\}. \tag{2.18}$$

For every r in $\mathrm{Iso}(D_{\underline{L}})$, d in $\mathbb{N}$ and (D, x) in $\mathrm{supp}(\underline{L})$, define a quadratic polynomial $Q_{r,D,x,d} : L \to \mathbb{Z}$ by the formula

$$Q_{r,D,x,d}(\lambda) := \beta(\lambda + dr + x) - D$$

and set $Q_{D,x} := Q_{0,D,x,d}$. Furthermore, define

$$\overline{d}_b := \begin{cases} d^{-1} \bmod b, & \text{if } (d,b) = 1 \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

The following holds:

Lemma 2.9. *If* $r \in \mathrm{Iso}(D_{\underline{L}})$ *and* $(D, x) \in \mathrm{supp}(\underline{L})$ *such that* $D > 0$*, then*

$$\begin{aligned} \sum_{c\geq 1} c^{-k} H_{\underline{L},c}(r, D, x) = &\sum_{b\geq 1} b^{-k} \sum_{d\in\mathbb{Z}_{(b)}} \sum_{\lambda\in L/bL} e_b\left(dQ_{r,D,x,\overline{d}_b}(\lambda)\right) \\ &+ \left(\frac{1}{\zeta(k - \mathrm{rk}(\underline{L}))} - 1\right) \sum_{b\geq 1} \frac{R_b(Q_{D,x})}{b^{k-1}}. \end{aligned} \tag{2.19}$$

Proof. Set $\lambda' := d^{-1}\lambda$ on the right-hand side of (2.16). Since $(d, c) = 1$, this change of variables is an automorphism of L/cL. Change the notation of the pair (D', r') to (D, x) and drop the prime from the new summation over λ' for simplicity. Use the fact that $r \in \mathrm{Iso}(D_{\underline{L}})$ in order to write $e_c(\beta(\lambda + r)) = e_c(\beta(\lambda + d^{-1}dr))$ and $e_c(\beta(\lambda, r)) = e_c(\beta(\lambda, d^{-1}dr))$. We obtain that

$$H_{\underline{L},c}(r, D, x) = \sum_{d\in\mathbb{Z}^\times_{(c)},\lambda\in L/cL} e_c\left(d\left(\beta(\lambda + d^{-1}r + x) - D\right)\right)$$

(note that this method would not work for $H_{\underline{L},c}(D, r, D', r')$). Remove the coprimality condition between d and c using (1.4):

$$H_{\underline{L},c}(r, D, x) = \sum_{d\in\mathbb{Z}_{(c)},\lambda\in L/cL} \sum_{a|(d,c)} \mu(a) e_c\left(d\left(\beta(\lambda + \overline{d}_c r + x) - D\right)\right).$$

Write $c = ab$ in order to obtain that

$$\sum_{c\geq 1} c^{-k} H_{\underline{L},c}(r, D, x) = \sum_{a\geq 1}\sum_{b\geq 1} \frac{\mu(a)}{(ab)^k} \sum_{\substack{d\in\mathbb{Z}_{(ab)} \\ a|d}} \sum_{\lambda\in L/abL} e_{ab}\left(d\left(\beta(\lambda + \overline{d}_{ab} r + x) - D\right)\right). \tag{2.20}$$

The condition that $d \in \mathbb{Z}_{(ab)}$ and $a \mid d$ is equivalent to the condition that $\frac{d}{a} \in \mathbb{Z}_{(b)}$. The expression $e_b\left(\frac{d}{a}\left(\beta(\lambda + \overline{d}_{ab} r + x) - D\right)\right)$ only depends on λ modulo b and therefore

$$\sum_{\lambda\in L/abL} e_b\left(\frac{d}{a}\left(\beta(\lambda + \overline{d}_{ab} r + x) - D\right)\right) = a^{\mathrm{rk}(\underline{L})} \sum_{\lambda\in L/bL} e_b\left(\frac{d}{a}\left(\beta(\lambda + \overline{d}_{ab} r + x) - D\right)\right).$$

Separate the term with $a = 1$ from the rest in (2.20) and note that $\overline{d}_{ab} = 0$ in the sum over $a \geq 2$, since $a \mid d$. Furthermore, substitute d for $\frac{d}{a}$ in the latter sum by abuse of notation in order to obtain that

$$\sum_{c\geq 1} c^{-k} H_{\underline{L},c}(r,D,x) = \sum_{b\geq 1} b^{-k} \sum_{d\in\mathbb{Z}_{(b)}} \sum_{\lambda\in L/bL} e_b\left(dQ_{r,D,x,\bar{d}_b}(\lambda)\right) + \sum_{a\geq 2} \frac{\mu(a)}{a^{k-\mathrm{rk}(\underline{L})}} \sum_{b\geq 1} b^{-k} \sum_{d\in\mathbb{Z}_{(b)}} \sum_{\lambda\in L/bL} e_b\left(dQ_{D,x}(\lambda)\right).$$

The following identity holds:

$$\frac{1}{b} \sum_{\lambda\in L/bL} \sum_{d\in\mathbb{Z}_{(b)}} e_b(dQ_{D,x}(\lambda)) = R_b(Q_{D,x}). \tag{2.21}$$

To check that this is true, assume that λ in L/bL satisfies $Q_{D,x}(\lambda) \equiv 0 \bmod b$. Then $e_b(dQ_{D,x}(\lambda)) = 1$ and, as d runs through congruence classes modulo b, we obtain a contribution of $|\mathbb{Z}_{(b)}|$ on the left-hand side of (2.21) from every such λ. On the other hand, if λ is such that $b \nmid Q_{D,x}(\lambda)$, then set $M := (b, Q_{D,x}(\lambda))$. The fact that $\sum_{d\in\mathbb{Z}_{(n)}} e_n(dm) = 0$ when $(n,m) = 1$ implies that

$$\sum_{d\in\mathbb{Z}_{(b)}} e_b(dQ_{D,x}(\lambda)) = M \sum_{d\in\mathbb{Z}_{\left(\frac{b}{M}\right)}} e_{\frac{b}{M}}\left(d\frac{Q_{D,x}(\lambda)}{M}\right) = 0.$$

In order to complete the proof of the lemma, combine (2.21) with the following well-known identity involving the Riemann zeta function:

$$\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s}.$$

□

2.3. Fourier coefficients of trivial Eisenstein series

Lemma 2.9 implies that Theorem 2.6, (*ii*) can be simplified in the following way when $r = 0$:

LEMMA 2.10. *The Eisenstein series $E_{k,\underline{L},0}$ vanishes identically when k is odd. When k is even, it has the following Fourier expansion:*

$$E_{k,\underline{L},0}(\tau,z) = \vartheta_{\underline{L},0}(\tau,z) + \sum_{\substack{(D,x)\in supp(\underline{L})\\ D<0}} G_{k,\underline{L},0}(D,x) e\left((\beta(x) - D)\tau + \beta(x,z)\right),$$

where

$$G_{k,\underline{L},0}(D,x) = \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right) \zeta(k - \mathrm{rk}(\underline{L}))} \sum_{b\geq 1} \frac{R_b(Q_{D,x})}{b^{k-1}}. \tag{2.22}$$

PROOF. It is clear from (2.17) that $C_0(E_{k,\underline{L},0})$ is equal to zero if k is odd and to $\vartheta_{\underline{L},0}$ otherwise. Set $r = 0$ in (2.19) and substitute this formula in (2.15):

$$G_{k,\underline{L},0}(D,x) = \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{2\det(\underline{L})^{-\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right) \zeta(k - \mathrm{rk}(\underline{L}))} \sum_{b\geq 1} \left(1 + (-1)^k\right) \frac{R_b(Q_{D,x})}{b^{k-1}}.$$

The result follows. □

An element λ in L/bL solves the modular equation

$$Q_{D,x}(\lambda) \equiv 0 \bmod b$$

if and only if $(-\lambda)$ solves the modular equation

$$Q_{D,-x}(-\lambda) \equiv 0 \bmod b.$$

It follows that $R_b(Q_{D,x}) = R_b(Q_{D,-x})$ and hence, in the notation of [**BK01**], $R_b(Q_{D,x}) = N_{x,-D}(b)$ and the Dirichlet series

$$L(s) := \sum_{b\geq 1} b^{-s} R_b(Q_{D,x})$$

is equal to $L_{x,-D}\left(s - \frac{\mathrm{rk}(\underline{L})}{2} + 1\right)$. This L-series converges for $\Re(s) > \mathrm{rk}(\underline{L})$ and it can be continued meromorphically to $\Re(s) > \frac{\mathrm{rk}(\underline{L})}{2} + 1$, with a simple pole at $s = \mathrm{rk}(\underline{L})$.

In order to compute $L(s)$, analyse the representation numbers

$$R_b := R_b(Q_{D,x})$$

in more detail. Set $\tilde{x} := N_x x$ and $\tilde{D} := N_x^2 D$. Then $\tilde{x} \in L$ and $\tilde{D} \in \mathbb{Z}$ (since $D \equiv \beta(x) \bmod \mathbb{Z}$). Assume that $(b, \det(\underline{L})) = 1$. Then $(b, N_x) = 1$ and therefore

$$\beta(\lambda + x) - D \equiv 0 \bmod b \iff \beta(N_x\lambda + \tilde{x}) \equiv \tilde{D} \bmod b.$$

The sets $\{N_x\lambda + \tilde{x} : \lambda \in L/bL\}$ and L/bL are in bijection. In other words, the following holds when $(b, N_x) = 1$:

$$R_b = \#\{\lambda \in L/bL : \beta(\lambda) \equiv \tilde{D} \bmod b\}. \tag{2.23}$$

For arbitrary b in $\mathbb{N}$, we have

$$\beta(\lambda + x) - D \equiv 0 \bmod b \iff \beta(N_x\lambda + \tilde{x}) \equiv \tilde{D} \bmod N_x^2 b$$

and $(N_x\lambda + \tilde{x})$ runs through representatives of the set

$$L/N_x bL \cap \{\lambda \in L : \lambda \equiv \tilde{x} \bmod N_x L\}$$

as λ runs through L/bL. In other words,

$$R_b = \#\{\lambda \in L/N_x bL : \beta(\lambda) \equiv \tilde{D} \bmod N_x^2 b \text{ and } \lambda \equiv \tilde{x} \bmod N_x L\}.$$

The representation numbers R_b satisfy the following:

Lemma 2.11. *The arithmetic function $b \mapsto R_b$ is multiplicative.*

Proof. Suppose that $b = b_1 b_2$. If λ in L/bL satisfies $Q_{D,x}(\lambda) \equiv 0 \bmod b$, then λ can be viewed as an element of L/b_1L such that $Q_{D,x}(\lambda) \equiv 0 \bmod b_1$ and it can also be viewed as an element of L/b_2L such that $Q_{D,x}(\lambda) \equiv 0 \bmod b_2$. Thus, the map defined by $\lambda \mapsto (\lambda \bmod b_1L, \lambda \bmod b_2L)$ gives the following embedding:

$$\begin{gathered}\{\lambda \in L/bL : b \mid Q_{D,x}(\lambda)\} \hookrightarrow \\ \{\lambda \in L/b_1L : b_1 \mid Q_{D,x}(\lambda)\} \times \{\lambda \in L/b_2L : b_2 \mid Q_{D,x}(\lambda)\}.\end{gathered} \tag{2.24}$$

Conversely, suppose that $(b_1, b_2) = 1$. Given λ_1 in L/b_1L satisfying $b_1 \mid Q_{D,x}(\lambda_1)$ and λ_2 in L/b_2L satisfying $b_2 \mid Q_{D,x}(\lambda_2)$, let $\lambda = (\lambda^1, \ldots, \lambda^{\mathrm{rk}(\underline{L})})^t$ be the lattice element whose coordinates are the unique solutions modulo b_1b_2 to the $\mathrm{rk}(\underline{L})$ systems of modular equations

$$\begin{cases} x \equiv \lambda_1^i \bmod b_1 & \text{and} \\ x \equiv \lambda_2^i \bmod b_2 \end{cases}$$

given by the Chinese remainder theorem. Then $\lambda \equiv \lambda_1 \bmod b_1L$ and $\lambda \equiv \lambda_2 \bmod b_2L$ and therefore $Q_{D,x}(\lambda) \equiv 0 \bmod b_1$ and $Q_{D,x}(\lambda) \equiv 0 \bmod b_2$. Since $(b_1, b_2) = 1$, this implies that $Q_{D,x}(\lambda) \equiv 0 \bmod b_1b_2$. In other words,

$$\{\lambda \in L/b_1L : b_1 \mid Q_{D,x}(\lambda)\} \times \{\lambda \in L/b_2L : b_2 \mid Q_{D,x}(\lambda)\} \hookrightarrow \{\lambda \in L/bL : b \mid Q_{D,x}(\lambda)\}$$

and, in view of (2.24), equality holds between these two sets. In particular, they have the same number of elements. ∎

It follows from Lemma 2.11 that $L(s)$ can be written as an Euler product:

$$L(s) = \prod_{p \text{ prime}} L_p(s),$$

where

$$L_p(s) = \sum_{l=0}^{\infty} p^{-ls} R_{p^l}. \tag{2.25}$$

Lemma 2.10 implies that the Fourier coefficients $G_{k,\underline{L},0}(D,x)$ are the values of the analytic continuation of

$$\frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right) \zeta(s - \mathrm{rk}(\underline{L}) + 1)} \prod_{p \text{ prime}} L_p(s)$$

at $s = k-1$. Before we state the following result from [**Sie35**] on representation numbers of quadratic forms modulo prime powers, note that

$$\beta(x) - D = n \iff \beta(N_x x, x) - 2N_x D = 2N_x n$$

and $2N_x n$ and $\beta(N_x x, x)$ are integers, implying that $2N_x D$ is an integer.

Lemma ([**BK01**, Lemma 5]). *Let p be a prime and set $w_p := 1 + 2v_p(2N_x D)$. If $l \geq w_p$, then the following recurrence relation holds:*

$$R_{p^{l+1}} = p^{\mathrm{rk}(\underline{L})-1} R_{p^l}.$$

In other words, we have $R_{p^{w_p+l}} = p^{l(\mathrm{rk}(\underline{L})-1)} R_{p^{w_p}}$ and therefore

$$L_p(s) = \sum_{l=0}^{w_p-1} p^{-ls} R_{p^l} + \frac{p^{-w_p s} R_{p^{w_p}}}{1 - p^{-(s-\mathrm{rk}(\underline{L})+1)}}.$$

Define the local Euler factor

$$\tilde{L}_p(s) := \left(1 - p^{-(s-\mathrm{rk}(\underline{L})+1)}\right) L_p(s) \tag{2.26}$$

and note that $\tilde{L}_p(s) = L_{x,-D}\left(s - \frac{\mathrm{rk}(\underline{L})}{2} + 1, p\right)$ in the notation of [**BK01**]. Then

$$L(s) = \zeta(s - \mathrm{rk}(\underline{L}) + 1) \prod_{p \text{ prime}} \tilde{L}_p(s). \tag{2.27}$$

We remind the reader of Definition 1.10 of $\chi_{\underline{L}}$. To compute the $\tilde{L}_p$'s, use the following result:

Theorem 2.12 ([**Sie35**, Hilfssatz 16]). *Let p be a prime which is coprime to $2\det(\underline{L})$ and set $\kappa := v_p(D)$. If l in $\mathbb{Z}$ is such that $l > \kappa$, the following holds:*

(i) If $\mathrm{rk}(\underline{L})$ is even, then

$$p^{l(1-\mathrm{rk}(\underline{L}))} R_{p^l} = \left(1 + \chi_{\underline{L}}(p) p^{1-\frac{\mathrm{rk}(\underline{L})}{2}} + \cdots + \chi_{\underline{L}}(p^\kappa) p^{\kappa(1-\frac{\mathrm{rk}(\underline{L})}{2})}\right)\left(1 - \chi_{\underline{L}}(p) p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right).$$

(ii) If $\mathrm{rk}(\underline{L})$ is odd, then write $D = D_0 f^2$, with D_0 in $\mathbb{Q}_{<0}$ and f in $\mathbb{N}$ such that $(f, 2\det(\underline{L})) = 1$ and $v_\ell(D_0) \in \{0,1\}$ for all primes ℓ which are coprime to $2\det(\underline{L})$ and set $\tilde{D}_0 := N_x^2 D_0$. If $\mathrm{rk}(\underline{L}) = 1$, then

$$R_{p^l} = \left(\chi_{\underline{L}}(\tilde{D}_0, p) + \chi_{\underline{L}}(\tilde{D}_0, p)^2\right) p^{v_p(f)}$$

and, if $\mathrm{rk}(\underline{L}) \geq 3$*, then*

$$p^{l(1-\mathrm{rk}(\underline{L}))}R_{p^l} = \left(\sigma_{2-\mathrm{rk}(\underline{L})}\left(p^{v_p(f)}\right) - \chi_{\underline{L}}(\tilde{D}_0, p)p^{-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor}\sigma_{2-\mathrm{rk}(\underline{L})}\left(p^{v_p(f)-1}\right)\right) \times \frac{1-p^{1-\mathrm{rk}(\underline{L})}}{1-\chi_{\underline{L}}(\tilde{D}_0, p)p^{-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor}}.$$

If the rank of $\underline{L}$ is odd, then the decomposition of D into $D = D_0 f^2$ can be reformulated in an explicit way. Write $D = \frac{A}{B}$, with A and B coprime integers, $A > 0$ and $B < 0$. Since $N_x^2 D$ is an integer, the prime factors of B are among the prime factors of N_x. Define

$$f := \prod_{\substack{p\mid A\\ p\nmid 2\det(\underline{L})}} p^{\lfloor\frac{v_p(A)}{2}\rfloor} \quad \text{and} \quad D_0 := \frac{D}{f^2} = \frac{1}{B}\prod_{\substack{p\mid A\\ p\mid 2\det(\underline{L})}} p^{v_p(A)} \prod_{\substack{p\mid A\\ p\nmid 2\det(\underline{L})}} p^{e_p}, \tag{2.28}$$

where, for every prime p,

$$e_p := \begin{cases} 1, & \text{if } v_p(A) \equiv 1 \bmod 2 \text{ and}\\ 0, & \text{otherwise.}\end{cases}$$

Then f and D_0 satisfy the conditions of Theorem 2.12. This theorem implies the following result:

Lemma 2.13. *Let* $\tilde{L}_p(\cdot)$ *denote the Euler factor* (2.26)*. If* $\mathrm{rk}(\underline{L})$ *is even, then*

$$\prod_p \tilde{L}_p(s) = \frac{1}{L\left(s - \frac{\mathrm{rk}(\underline{L})}{2} + 1, \chi_{\underline{L}}\right)} \prod_{p\mid 2\tilde{D}\det(\underline{L})} \frac{\tilde{L}_p(s)}{1-\chi_{\underline{L}}(p)p^{-(s-\frac{\mathrm{rk}(\underline{L})}{2}+1)}}$$

and if $\mathrm{rk}(\underline{L})$ *is odd, then*

$$\prod_p \tilde{L}_p(s) = \frac{L\left(s-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor, \chi_{\underline{L}}(\tilde{D}_0, \cdot)\right)}{\zeta(2s - \mathrm{rk}(\underline{L})+1)} \prod_{p\mid\tilde{D}\det(\underline{L})} \frac{1-\chi_{\underline{L}}(\tilde{D}_0, p)p^{-\left(s-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor\right)}}{1-p^{-(2s-\mathrm{rk}(\underline{L})+1)}}\tilde{L}_p(s).$$

Proof. We remind the reader that $\tilde{D} = N_x^2 D$ and that $\tilde{D}_0 = N_x^2 D_0$. Therefore, if $(p, 2\tilde{D}\det(\underline{L})) = 1$, then $w_p = 1$ and equation (2.26) can be reformulated as

$$\tilde{L}_p(s) = 1 - p^{-(s-\mathrm{rk}(\underline{L})+1)} + p^{-s}R_p.$$

If $\mathrm{rk}(\underline{L})$ is even, then Theorem 2.12, (i) implies that

$$R_p = p^{\mathrm{rk}(\underline{L})-1}(1-\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}})$$

for every prime p which is coprime to $2\tilde{D}\det(\underline{L})$ and hence

$$\prod_p \tilde{L}_p(s) = \prod_{p\nmid 2\tilde{D}\det(\underline{L})}\left(1-\chi_{\underline{L}}(p)p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)\prod_{p\mid 2\tilde{D}\det(\underline{L})}\tilde{L}_p.$$

If $\mathrm{rk}(\underline{L})$ is odd and p is coprime to $2\tilde{D}\det(\underline{L})$, then write $D = D_0 f^2$ with the required conditions and note that $v_p(f) = 0$ (since $\tilde{D}$ is a multiple of f) and $\chi_{\underline{L}}(\tilde{D}_0, p) \in \{\pm 1\}$. If $\mathrm{rk}(\underline{L}) = 1$, then Theorem 2.12, (ii) implies that $R_p = \chi_{\underline{L}}(\tilde{D}_0, p) + 1$ and, if $\mathrm{rk}(\underline{L}) \geq 3$, then

$$p^{1-\mathrm{rk}(\underline{L})}R_p = \frac{1-p^{1-\mathrm{rk}(\underline{L})}}{1-\chi_{\underline{L}}(\tilde{D}_0, p)p^{-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor}} = 1 + \chi_{\underline{L}}(\tilde{D}_0, p)p^{-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor}.$$

Thus,

$$\tilde{L}_p(s) = 1 + \chi_{\underline{L}}(\tilde{D}_0, p)p^{-(s-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor)}$$

and hence

$$\prod_p \tilde{L}_p(s) = \prod_{p\nmid 2\tilde{D}\det(\underline{L})} \frac{1-p^{-(2s-\mathrm{rk}(\underline{L})+1)}}{1-\chi_{\underline{L}}(\tilde{D}_0,p)p^{-\left(s-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor\right)}} \prod_{p\mid 2\tilde{D}\det(\underline{L})} \tilde{L}_p(s),$$

completing the proof. □

When $\mathrm{rk}(\underline{L})$ is even, write $\Delta(\underline{L}) = \mathfrak{d}\mathfrak{f}^2$, with $\mathfrak{d}$ the discriminant of the quadratic field $\mathbb{Q}(\sqrt{\Delta(\underline{L})})$ and $\mathfrak{f}$ in $\mathbb{N}$. When $\mathrm{rk}(\underline{L})$ is odd, for every pair (D,r) in the support of $\underline{L}$ such that $D<0$, set $\tilde{D}_{0,x} := D_0N_x^2$ and write $\tilde{D}_{0,x}\Delta(\underline{L}) = \mathfrak{d}_{D,x}\mathfrak{f}_{D,x}^2$, with $\mathfrak{d}_{D,x}$ the discriminant of the quadratic field $\mathbb{Q}(\sqrt{\tilde{D}_{0,x}\Delta(\underline{L})})$ and $\mathfrak{f}_{D,x}$ in $\mathbb{N}$. We remind the reader that $L_D(\cdot)$ denotes the Dirichlet L-function of the quadratic character χ_D for every discriminant D. The main result of this section is the following:

Theorem 2.14. *Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$ and let k be a positive, even integer such that $k > \frac{\mathrm{rk}(\underline{L})}{2}+2$. Let the pair (D,x) in* $\mathrm{supp}(\underline{L})$ *be such that $D<0$ and let $G_{k,\underline{L},0}(\cdot,\cdot)$ denote the non-singular Fourier coefficients of $E_{k,\underline{L},0}$. If* $\mathrm{rk}(\underline{L})$ *is even, then*

$$G_{k,\underline{L},0}(D,x) = \frac{2(-1)^{\lceil\frac{\mathrm{rk}(\underline{L})}{4}\rceil}(-D|\mathfrak{d}|)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\mathfrak{f}L_{\mathfrak{d}}\left(1-k+\frac{\mathrm{rk}(\underline{L})}{2}\right)\sum_{d\mid\mathfrak{f}}\mu(d)\chi_{\mathfrak{d}}(d)d^{\frac{\mathrm{rk}(\underline{L})}{2}-k}\sigma_{1-2k+\mathrm{rk}(\underline{L})}\left(\frac{\mathfrak{f}}{d}\right)} \times \prod_{p\mid 2\tilde{D}\det(\underline{L})}\frac{\tilde{L}_p(k-1)}{1-\chi_{\underline{L}}(p)p^{\frac{\mathrm{rk}(\underline{L})}{2}-k}} \tag{2.29}$$

and if, $\mathrm{rk}(\underline{L})$ *is odd, then*

$$\begin{aligned} G_{k,\underline{L},0}(D,x) =& \frac{\chi_8(\mathrm{rk}(\underline{L}))\,2^{2k-\mathrm{rk}(\underline{L})}\left(\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-k\right)(D\tilde{D}_{0,x})^{\frac{1}{2}}(-D)^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}}{B_{2k-\mathrm{rk}(\underline{L})-1}\mathfrak{f}_{D,x}|\mathfrak{d}_{D,x}|^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}} \\ &\times L_{\mathfrak{d}_{D,x}}\left(1-k+\lceil\tfrac{\mathrm{rk}(\underline{L})}{2}\rceil\right)\sum_{d\mid\mathfrak{f}_{D,x}}\mu(d)\chi_{\mathfrak{d}_{D,x}}(d)d^{\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-k} \\ &\times\sigma_{2-2k+\mathrm{rk}(\underline{L})}\left(\frac{\mathfrak{f}_{D,x}}{d}\right)\prod_{p\mid\tilde{D}\det(\underline{L})}\frac{1-\chi_{\underline{L}}(\tilde{D}_0,p)p^{\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-k}}{1-p^{1-2k+\mathrm{rk}(\underline{L})}}\tilde{L}_p(k-1). \end{aligned} \tag{2.30}$$

Proof. Equations (2.22) and (2.27) and Lemma 2.13 imply that

$$G_{k,\underline{L},0}(D,x) = \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}}i^k(-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\det(\underline{L})^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)L\left(k-\frac{\mathrm{rk}(\underline{L})}{2},\chi_{\underline{L}}(1,\cdot)\right)}\prod_{p\mid 2\tilde{D}\det(\underline{L})}\frac{\tilde{L}_p(k-1)}{1-\chi_{\underline{L}}(p)p^{-(k-\frac{\mathrm{rk}(\underline{L})}{2})}}$$

if $\mathrm{rk}(\underline{L})$ is even and that

$$\begin{aligned} G_{k,\underline{L},0}(D,x) =& \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}}i^k(-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}L\left(k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil,\chi_{\underline{L}}(\tilde{D}_{0,x},\cdot)\right)}{\det(\underline{L})^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)\zeta(2k-\mathrm{rk}(\underline{L})-1)} \\ &\times\prod_{p\mid\tilde{D}\det(\underline{L})}\frac{1-\chi_{\underline{L}}(\tilde{D}_{0,x},p)p^{-\left(k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil\right)}}{1-p^{-(2k-\mathrm{rk}(\underline{L})-1)}}\tilde{L}_p(k-1) \end{aligned}$$

if $\mathrm{rk}(\underline{L})$ is odd.

Suppose that $\mathrm{rk}(\underline{L})$ is even and that the pair (D,x) in the support of $\underline{L}$ $(D<0)$ is fixed. Then $\Delta(\underline{L})$ is a discriminant, i.e. it is congruent to 0 or 1 modulo 4, and hence $\chi_{\underline{L}}(\cdot) = \left(\frac{\Delta(\underline{L})}{\cdot}\right)$ is a quadratic character modulo $|\Delta(\underline{L})|$. Write $\Delta(\underline{L}) = \mathfrak{d}\mathfrak{f}^2$, with $\mathfrak{d}$ the discriminant of $\mathbb{Q}(\sqrt{\Delta(\underline{L})})$ and $\mathfrak{f}$ in $\mathbb{N}$. It follows that $\chi_{\mathfrak{d}}(\cdot) = \left(\frac{\mathfrak{d}}{\cdot}\right)$ is a primitive quadratic character modulo $|\mathfrak{d}|$. It was shown in [**Zag77**, §4] that the Dirichlet L-function of $\chi_{\underline{L}}$ satisfies the following:

$$L(s, \chi_{\underline{L}}) = L_{\mathfrak{d}}(s) \sum_{d|\mathfrak{f}} \mu(d) \left(\frac{\mathfrak{d}}{d}\right) d^{-s} \sigma_{1-2s}\left(\frac{\mathfrak{f}}{d}\right). \tag{2.31}$$

Write $G_{k,\underline{L},0}(D, x) = A \times B$, with

$$A := \frac{2^{k-\frac{\mathrm{rk}(\underline{L})}{2}} i^k (-D)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\sum_{d|\mathfrak{f}} \mu(d) \left(\frac{\mathfrak{d}}{d}\right) d^{\frac{\mathrm{rk}(\underline{L})}{2}-k} \sigma_{1+\mathrm{rk}(\underline{L})-2k}\left(\frac{\mathfrak{f}}{d}\right)} \prod_{p|2\tilde{D}\det(\underline{L})} \frac{\tilde{L}_p(k-1)}{1-\chi_{\underline{L}}(p)p^{-(k-\frac{\mathrm{rk}(\underline{L})}{2})}}$$

and

$$B := \frac{\pi^{k-\frac{\mathrm{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right) L_{\mathfrak{d}}\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)}.$$

Since k and $\mathrm{rk}(\underline{L})$ are even, it follows that A is a rational number. Re-write the expression for B using functional equations of Dirichlet L-series (1.5):

$$\begin{aligned}
B =& \frac{\pi^{k-\frac{\mathrm{rk}(\underline{L})}{2}} \left(\frac{|\mathfrak{d}|}{\pi}\right)^{\frac{k-\mathrm{rk}(\underline{L})/2+a_{\chi_{\mathfrak{d}}}}{2}} \Gamma\left(\frac{k-\mathrm{rk}(\underline{L})/2+a_{\chi_{\mathfrak{d}}}}{2}\right)}{\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right) \Lambda\left(k - \frac{\mathrm{rk}(\underline{L})}{2}, \chi_{\mathfrak{d}}\right)} \\
=& \frac{i^{a_{\chi_{\mathfrak{d}}}} |\mathfrak{d}|^{\frac{1}{2}} \pi^{\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} - \frac{a_{\chi_{\mathfrak{d}}}}{2}} |\mathfrak{d}|^{\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} + \frac{a_{\chi_{\mathfrak{d}}}}{2}} \Gamma\left(\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} + \frac{a_{\chi_{\mathfrak{d}}}}{2}\right)}{\det(\underline{L})^{\frac{1}{2}} \Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right) G(\chi_{\mathfrak{d}}) \Lambda\left(1 - k + \frac{\mathrm{rk}(\underline{L})}{2}, \overline{\chi}_{\mathfrak{d}}\right)} \\
=& \frac{i^{a_{\chi_{\mathfrak{d}}}} \pi^{\frac{1}{2}} |\mathfrak{d}|^{k-\frac{\mathrm{rk}(\underline{L})}{2}}}{\det(\underline{L})^{\frac{1}{2}} G(\chi_{\mathfrak{d}}) L_{\mathfrak{d}}\left(1 - k + \frac{\mathrm{rk}(\underline{L})}{2}\right)} \cdot \frac{\Gamma\left(\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} + \frac{a_{\chi_{\mathfrak{d}}}}{2}\right)}{\Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right) \Gamma\left(\frac{1+a_{\chi_{\mathfrak{d}}}}{2} - \frac{k}{2} + \frac{\mathrm{rk}(\underline{L})}{4}\right)},
\end{aligned}$$

since $\chi_{\mathfrak{d}}$ is a real character. We remind the reader that

$$\Delta(\underline{L}) = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \det(\underline{L}) = \mathfrak{d}\mathfrak{f}^2$$

and therefore $\mathfrak{d} > 0$ if $\mathrm{rk}(\underline{L}) \equiv 0 \bmod 4$ and $\mathfrak{d} < 0$ if $\mathrm{rk}(\underline{L}) \equiv 2 \bmod 4$. The Kronecker symbol $\left(\frac{n}{-1}\right)$ is equal to $\mathrm{sign}(n)$ and therefore

$$a_{\chi_{\mathfrak{d}}} = \begin{cases} 0, & \text{if } \mathrm{rk}(\underline{L}) \equiv 0 \bmod 4 \text{ and} \\ 1, & \text{if } \mathrm{rk}(\underline{L}) \equiv 2 \bmod 4. \end{cases}$$

The Gauss sum $G(\chi_{\mathfrak{d}})$ is equal to $\mathfrak{d}^{\frac{1}{2}}$, since $\mathfrak{d}$ is a fundamental discriminant (see [**CS17**, §3.4.2] for a proof, for example), and

$$\det(\underline{L})^{\frac{1}{2}} G(\chi_{\mathfrak{d}}) = (\det(\underline{L})\mathfrak{d})^{\frac{1}{2}} = ((-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \mathfrak{d}^2 \mathfrak{f}^2)^{\frac{1}{2}} = i^{a_{\chi_{\mathfrak{d}}}} |\mathfrak{d}| \mathfrak{f}.$$

The Gamma function satisfies the following duplication formula:

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \pi^{\frac{1}{2}} \Gamma(2z). \tag{2.32}$$

Thus,

$$\frac{\Gamma\left(\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} + \frac{a_{\chi_{\mathfrak{d}}}}{2}\right)}{\Gamma\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)} = \frac{2^{1-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)} \pi^{\frac{1}{2}}}{\Gamma\left(\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} + \frac{1-a_{\chi_{\mathfrak{d}}}}{2}\right)}.$$

The Gamma function satisfies Euler's reflection formula:

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}, \quad z \notin \mathbb{Z}. \tag{2.33}$$

Furthermore, we have $\sin(\pi(n+1/2)) = (-1)^n$ for integer n. It follows that

$$\frac{\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}+\frac{a_{\chi_{\mathfrak{d}}}}{2}\right)}{\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)\Gamma\left(\frac{1+a_{\chi_{\mathfrak{d}}}}{2}-\frac{k}{2}+\frac{\mathrm{rk}(\underline{L})}{4}\right)} = \frac{2^{1-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}}{\pi^{\frac{1}{2}}}(-1)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})}{4}-\frac{a_{\chi_{\mathfrak{d}}}}{2}}.$$

Thus,

$$B = \frac{2^{1-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}(-1)^{\frac{k}{2}-\lceil\frac{\mathrm{rk}(\underline{L})}{4}\rceil}|\mathfrak{d}|^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}}{\mathfrak{f}L_{\mathfrak{d}}\left(1-k+\frac{\mathrm{rk}(\underline{L})}{2}\right)}$$

when $\mathrm{rk}(\underline{L})$ is even.

Suppose that $\mathrm{rk}(\underline{L})$ is odd. Then $\Delta(\underline{L})$ is congruent to 0 modulo 4 (see Remark 1.8) and hence $\chi_{\underline{L}}(\tilde{D}_0,\cdot) := \left(\frac{\tilde{D}_0\Delta(\underline{L})}{\cdot}\right)$ is a quadratic character modulo $|\tilde{D}_0\Delta(\underline{L})|$. Write $\tilde{D}_0\Delta(\underline{L}) = \mathfrak{d}\mathfrak{f}^2$, with $\mathfrak{d}$ the discriminant of $\mathbb{Q}(\sqrt{\tilde{D}_0\Delta(\underline{L})})$ and $\mathfrak{f}$ in $\mathbb{N}$. As before, the L-function of $\chi_{\underline{L}}(\tilde{D}_0,\cdot)$ satisfies the following:

$$L(s,\chi_{\underline{L}}(\tilde{D}_0,\cdot)) = L_{\mathfrak{d}}(s)\sum_{d|\mathfrak{f}}\mu(d)\left(\tfrac{\mathfrak{d}}{d}\right)d^{-s}\sigma_{1-2s}\left(\frac{\mathfrak{f}}{d}\right).$$

The values of the Riemann zeta function at positive even integers are well-known:

$$\zeta(2k-\mathrm{rk}(\underline{L})-1) = \frac{(-1)^{k-\frac{\mathrm{rk}(\underline{L})-1}{2}}B_{2k-\mathrm{rk}(\underline{L})-1}(2\pi)^{2k-\mathrm{rk}(\underline{L})-1}}{2\Gamma(2k-\mathrm{rk}(\underline{L}))}.$$

Write $G_{k,\underline{L},0}(D,x) = A\times B$, with

$$A := \frac{2i^k(-D)^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\sum_{d|\mathfrak{f}}\mu(d)\left(\tfrac{\mathfrak{d}}{d}\right)d^{\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-k}\sigma_{2+\mathrm{rk}(\underline{L})-2k}\left(\tfrac{\mathfrak{f}}{d}\right)}{(-1)^{k-\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor}B_{2k-\mathrm{rk}(\underline{L})-1}}$$
$$\times\prod_{p|\tilde{D}\det(\underline{L})}\frac{1-\chi_{\underline{L}}(\tilde{D}_0,p)p^{-\left(k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil\right)}}{1-p^{-(2k-\mathrm{rk}(\underline{L})-1)}}\tilde{L}_p(k-1)$$

and note that A is a rational number. Applying the functional equation (1.5) to $\chi_{\mathfrak{d}}$ yields

$$\begin{aligned}B =& \frac{(2\pi)^{k-\frac{\mathrm{rk}(\underline{L})}{2}}(-D)^{\frac{1}{2}}\Gamma\left(2k-\mathrm{rk}(\underline{L})\right)L_{\mathfrak{d}}\left(k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil\right)}{\det(\underline{L})^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)(2\pi)^{2k-\mathrm{rk}(\underline{L})-1}}\\ =& \frac{(-D)^{\frac{1}{2}}G(\chi_{\mathfrak{d}})L_{\mathfrak{d}}\left(1-k+\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil\right)}{i^{a_{\chi_{\mathfrak{d}}}}\det(\underline{L})^{\frac{1}{2}}2^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}|\mathfrak{d}|^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}}\cdot\frac{\Gamma\left(\frac{1+a_{\chi_{\mathfrak{d}}}}{2}-\frac{k}{2}+\frac{\mathrm{rk}(\underline{L})+1}{4}\right)\Gamma\left(2k-\mathrm{rk}(\underline{L})\right)}{\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})+1}{4}+\frac{a_{\chi_{\mathfrak{d}}}}{2}\right)}.\end{aligned}$$

We remind the reader that

$$\tilde{D}_0\Delta(\underline{L}) = \tilde{D}_0(-1)^{\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor}2\det(\underline{L}) = \mathfrak{d}\mathfrak{f}^2$$

and $\tilde{D}_0 < 0$ and therefore $\mathfrak{d} > 0$ if $\mathrm{rk}(\underline{L}) \equiv 3 \bmod 4$ and $\mathfrak{d} < 0$ if $\mathrm{rk}(\underline{L}) \equiv 1 \bmod 4$. It follows that

$$a_{\chi_{\mathfrak{d}}} = \begin{cases}0, & \text{if } \mathrm{rk}(\underline{L}) \equiv 3 \bmod 4 \text{ and}\\ 1, & \text{if } \mathrm{rk}(\underline{L}) \equiv 1 \bmod 4.\end{cases}$$

The Gauss sum $G(\chi_{\mathfrak{d}})$ is equal to $\mathfrak{d}^{\frac{1}{2}}$ and

$$\frac{G(\chi_{\mathfrak{d}})}{\det(\underline{L})^{\frac{1}{2}}} = \left(\frac{\mathfrak{d}}{\det(\underline{L})}\right)^{\frac{1}{2}} = \left(\frac{(-1)^{\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor}2\tilde{D}_0}{\mathfrak{f}^2}\right)^{\frac{1}{2}} = \frac{i^{a_{\chi_{\mathfrak{d}}}}(-\tilde{D}_0)^{\frac{1}{2}}2^{\frac{1}{2}}}{\mathfrak{f}}.$$

The duplication formula (2.32) implies that

$$\frac{\Gamma\left(2k-\mathrm{rk}(\underline{L})\right)}{\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}=\frac{\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})-1}{2}\right)}{2^{1-(2k-\mathrm{rk}(\underline{L}))}\pi^{\frac{1}{2}}}.$$

Euler's reflection formula (2.33) implies that

$$\begin{aligned}\Gamma\left(\frac{1+a_{\chi_{\mathfrak{d}}}}{2}-\frac{k}{2}+\frac{\mathrm{rk}(\underline{L})+1}{4}\right)&=\Gamma\left(1-\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})-1+2a_{\chi_{\mathfrak{d}}}}{4}\right)\right)\\&=\frac{\pi}{\sin\left(\pi\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})-1+2a_{\chi_{\mathfrak{d}}}}{4}\right)\right)\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})-1+2a_{\chi_{\mathfrak{d}}}}{4}\right)}\\&=\frac{\pi}{(-1)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})+1+2a_{\chi_{\mathfrak{d}}}}{4}}\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})-1+2a_{\chi_{\mathfrak{d}}}}{4}\right)}.\end{aligned}$$

Finally, using the duplication formula, we have

$$\begin{aligned}&\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})+1}{4}+\frac{a_{\chi_{\mathfrak{d}}}}{2}\right)\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})-1+2a_{\chi_{\mathfrak{d}}}}{4}\right)\\&=\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})+1}{4}\right)\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})-1}{4}\right)=2^{\frac{\mathrm{rk}(\underline{L})+3}{2}-k}\pi^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})+1}{2}\right)\end{aligned}$$

and it follows that

$$\begin{aligned}\frac{\Gamma\left(\frac{1+a_{\chi_{\mathfrak{d}}}}{2}-\frac{k}{2}+\frac{\mathrm{rk}(\underline{L})+1}{4}\right)\Gamma\left(2k-\mathrm{rk}(\underline{L})\right)}{\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)\Gamma\left(\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})+1}{4}+\frac{a_{\chi_{\mathfrak{d}}}}{2}\right)}&=\frac{\pi^{\frac{1}{2}}(-1)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})+1+2a_{\chi_{\mathfrak{d}}}}{4}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})-1}{2}\right)}{2^{1-(2k-\mathrm{rk}(\underline{L}))}2^{\frac{\mathrm{rk}(\underline{L})+3}{2}-k}\pi^{\frac{1}{2}}\Gamma\left(k-\frac{\mathrm{rk}(\underline{L})+1}{2}\right)}\\&=\frac{(-1)^{\frac{k}{2}-\frac{\mathrm{rk}(\underline{L})+1+2a_{\chi_{\mathfrak{d}}}}{4}}\left(k-\frac{\mathrm{rk}(\underline{L})+1}{2}\right)}{2^{\frac{5}{2}-3k+\frac{3\mathrm{rk}(\underline{L})}{2}}}.\end{aligned}$$

Thus,

$$B=\frac{(-1)^{\frac{k}{2}-\lceil\frac{\mathrm{rk}(\underline{L})}{4}\rceil}\left(k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil\right)(D\tilde{D}_0)^{\frac{1}{2}}L_{\mathfrak{d}}\left(1-k+\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil\right)}{\mathfrak{f}2^{\mathrm{rk}(\underline{L})-2k+1}|\mathfrak{d}|^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}}$$

when $\mathrm{rk}(\underline{L})$ is odd. Note that

$$(-1)^{\lfloor\frac{\mathrm{rk}(\underline{L})}{2}\rfloor-\lceil\frac{\mathrm{rk}(\underline{L})}{4}\rceil}=\begin{cases}1, & \text{if } \mathrm{rk}(\underline{L})\equiv\pm 3 \bmod 8 \text{ and}\\-1, & \text{if } \mathrm{rk}(\underline{L})\equiv\pm 1 \bmod 8\end{cases}$$

and we remind the reader that

$$\left(\frac{2}{p}\right)=\begin{cases}1, & \text{if } p\equiv\pm 1 \bmod 8 \text{ and}\\-1, & \text{if } p\equiv\pm 3 \bmod 8\end{cases}$$

for every odd prime p, which completes the proof. □

An important consequence of Theorem 2.14 is the following rationality result:

Corollary 2.15. *The Fourier coefficients of $E_{k,\underline{L},0}$ are rational numbers.*

Proof. The values of $L_{\mathfrak{d}}(\cdot)$ at negative integers can be expressed in terms of Bernoulli polynomials (1.3) (see [**Zag81**, §1.7] for a proof, for example):

$$L_{\mathfrak{d}}(-n)=-\frac{|\mathfrak{d}|^n}{n+1}\sum_{m=1}^{|\mathfrak{d}|}\chi_{\mathfrak{d}}(m)B_{n+1}\left(\frac{m}{|\mathfrak{d}|}\right). \tag{2.34}$$

Furthermore, when $\mathrm{rk}(\underline{L})$ is odd, we have

$$(D\tilde{D}_0)^{\frac{1}{2}} = \left(\frac{N_x^2 D^2}{f^2}\right)^{\frac{1}{2}} = \frac{-N_x D}{f}.$$

Every other quantity appearing in (2.29) and (2.30) is rational. The result follows. □

While the formulas obtained in Theorem 2.14 are explicit, we want to investigate the Euler factors $\tilde{L}_p$ at primes dividing $2\tilde{D}\det(\underline{L})$. First, compute the product expansion of the sum on the right-hand side of (2.31). The following holds:

Lemma 2.16. *If $\mathfrak{d}$ is a fundamental discriminant and s is a complex parameter, then the arithmetic function $F_s : \mathbb{N} \to \mathbb{C}$, defined by*

$$F_s(\mathfrak{f}) := \sum_{d \mid \mathfrak{f}} \mu(d)\chi_{\mathfrak{d}}(d) d^{-s} \sigma_{1-2s}\left(\frac{\mathfrak{f}}{d}\right),$$

has the following product expansion:

$$F_s(\mathfrak{f}) = \prod_{p \mid \mathfrak{f}} \left(1 + \left(1 - \chi_{\mathfrak{d}}(p) p^{s-1}\right) \frac{1 - p^{v_p(\mathfrak{f})(1-2s)}}{p^{2s-1} - 1}\right). \tag{2.35}$$

Proof. Assume that a and b in $\mathbb{N}$ are such that $(a, b) = 1$. Then every divisor d of ab can be written as $d = d_a d_b$, with $d_a \mid a$, $d_b \mid b$ and $(d_a, d_b) = \left(\frac{a}{d_a}, \frac{b}{d_b}\right) = 1$ and therefore

$$F_s(ab) = \sum_{d_a \mid a} \mu(d_a)\chi_{\mathfrak{d}}(d_a) d_a^{-s} \sigma_{1-2s}\left(\frac{a}{d_a}\right) \sum_{d_b \mid b} \mu(d_b)\chi_{\mathfrak{d}}(d_b) d_b^{-s} \sigma_{1-2s}\left(\frac{b}{d_b}\right) = F_s(a)F_s(b),$$

since $\mu(\cdot)$, $\chi_{\mathfrak{d}}(\cdot)$ and $\sigma_{1-2s}(\cdot)$ are multiplicative functions. It follows that F_s is multiplicative. In particular, it suffices to prove (2.35) for $\mathfrak{f}$ equal to a prime power, say $\mathfrak{f} = p^t$ for some prime p and some t in $\mathbb{N}$. We have

$$F_s(p^t) = \sigma_{1-2s}(p^t) - \chi_{\mathfrak{d}}(p) p^{-s} \sigma_{1-2s}(p^{t-1}).$$

Using the product expansion of the divisor sum, we obtain that

$$F_s(p^t) = \frac{p^{(t+1)(1-2s)} - 1}{p^{1-2s} - 1} - \chi_{\mathfrak{d}}(p) p^{-s} \frac{p^{t(1-2s)} - 1}{p^{1-2s} - 1} = 1 + \left(1 - \chi_{\mathfrak{d}}(p) p^{s-1}\right) \frac{1 - p^{t(1-2s)}}{p^{2s-1} - 1},$$

as claimed. □

When $p \mid \mathfrak{d}$, we have $\chi_{\mathfrak{d}}(p) = 0$ and Lemma 2.16 implies that

$$\begin{aligned} F_s(\mathfrak{f}) &= \prod_{p \mid \mathfrak{d}} \frac{p^{(v_p(\mathfrak{f})+1)(1-2s)} - 1}{p^{1-2s} - 1} \prod_{p \mid \mathfrak{f}, p \nmid \mathfrak{d}} \left(1 + \left(1 - \chi_{\mathfrak{d}}(p) p^{s-1}\right) \frac{1 - p^{v_p(\mathfrak{f})(1-2s)}}{p^{2s-1} - 1}\right) \\ &= \sigma_{1-2s}(\mathfrak{g}) \prod_{p \mid \mathfrak{f}, p \nmid \mathfrak{d}} \left(1 + \left(1 - \chi_{\mathfrak{d}}(p) p^{s-1}\right) \frac{1 - p^{v_p(\mathfrak{f})(1-2s)}}{p^{2s-1} - 1}\right), \end{aligned}$$

where $\mathfrak{g} = \prod_{p \mid \mathfrak{d}} p^{v_p(\mathfrak{f})}$.

Let $\mathfrak{d}$ and $\mathfrak{f}$ be defined as in the proof of Theorem 2.14. We compute their prime decomposition. If $\mathrm{rk}(\underline{L})$ is odd, then Definition 1.9 and equation (2.28) imply that

$$\tilde{D}_0 \Delta(\underline{L}) = N_x^2 D_0 (-1)^{\lfloor \frac{\mathrm{rk}(\underline{L})}{2} \rfloor} 2\det(\underline{L}) = (-1)^{\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil} 2^{v_2(2\tilde{D}\det(\underline{L}))} \prod_{\substack{p \mid \det(\underline{L}) \\ p \neq 2}} p^{v_p(\tilde{D}\det(\underline{L}))} \prod_{p \nmid \det(\underline{L})} p^{e_p}.$$

For every odd prime $p \mid \tilde{D}\det(\underline{L})$, define the constant

$$g_p := \begin{cases} 1, & \text{if } v_p(\tilde{D}\det(\underline{L})) \equiv 1 \bmod 2 \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

We remind the reader that a fundamental discriminant is an integer which is either congruent to 1 modulo 4 and square-free, or equal to $4n$, for some square-free n which is congruent to 2 or 3 modulo 4. It follows that the sign of $\tilde{D}_0\Delta(\underline{L})$ is important and we must distinguish between the cases where $\mathrm{rk}(\underline{L}) \equiv 1 \bmod 4$ and $\mathrm{rk}(\underline{L}) \equiv 3 \bmod 4$. Write $D = \frac{A}{B}$, with A and B coprime integers, $A > 0$ and $B < 0$ and set

$$S := \{p \equiv 3 \bmod 4 : (p \mid A, p \nmid \det(\underline{L}), e_p = 1) \text{ or } (p \mid \det(\underline{L}), g_p = 1)\}.$$

If $\mathrm{rk}(\underline{L}) \equiv 1 \bmod 4$, then $\tilde{D}_0\Delta(\underline{L})$ is negative and therefore

$$\mathfrak{d} = -\prod_{p\nmid\det(\underline{L})} p^{e_p} \prod_{\substack{p\mid\det(\underline{L})\\ p\neq 2}} p^{g_p} \times \begin{cases} 1, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ is even and } |S| \text{ is odd,} \\ 4, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ and } |S| \text{ are even and} \\ 8, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ is odd} \end{cases}$$

and

$$\mathfrak{f} = \prod_{\substack{p\mid\det(\underline{L})\\ p\neq 2}} p^{\lfloor \frac{v_p(\tilde{D}\det(\underline{L}))}{2} \rfloor} \times \begin{cases} 2^{\frac{v_2(2\tilde{D}\det(\underline{L}))}{2}}, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ even and } |S| \text{ odd,} \\ 2^{\frac{v_2(\tilde{D}\det(\underline{L})/2)}{2}}, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ and } |S| \text{ even and} \\ 2^{\frac{v_2(\tilde{D}\det(\underline{L})/4)}{2}}, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ is odd.} \end{cases}$$

On the other hand, if $\mathrm{rk}(\underline{L}) \equiv 3 \bmod 4$, then $\tilde{D}_0\Delta(\underline{L})$ is positive and therefore

$$\mathfrak{d} = \prod_{\substack{p\mid A\\ p\nmid\det(\underline{L})}} p^{e_p} \prod_{\substack{p\mid\det(\underline{L})\\ p\neq 2}} p^{g_p} \times \begin{cases} 1, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ and } |S| \text{ are even,} \\ 4, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ is even and } |S| \text{ is odd and} \\ 8, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ is odd} \end{cases}$$

and

$$\mathfrak{f} = \prod_{\substack{p\mid\det(\underline{L})\\ p\neq 2}} p^{\lfloor \frac{v_p(\tilde{D}\det(\underline{L}))}{2} \rfloor} \times \begin{cases} 2^{\frac{v_2(2\tilde{D}\det(\underline{L}))}{2}}, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ and } |S| \text{ are even,} \\ 2^{\frac{v_2(\tilde{D}\det(\underline{L})/2)}{2}}, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ even and } |S| \text{ odd and} \\ 2^{\frac{v_2(\tilde{D}\det(\underline{L})/4)}{2}}, & \text{if } v_2(2\tilde{D}\det(\underline{L})) \text{ is odd.} \end{cases}$$

It follows that, when $\mathrm{rk}(\underline{L})$ is odd, the primes that divide both $\mathfrak{d}$ and $\mathfrak{f}$ are those odd primes dividing $\det(\underline{L})$, which have odd valuation greater than or equal to 3 in $\tilde{D}\det(\underline{L})$, and possibly the prime 2. Primes that divide $\mathfrak{d}$, but not $\mathfrak{f}$, are odd primes dividing $\tilde{D}$, but not $\det(\underline{L})$, which have odd valuation in A, odd primes dividing $\det(\underline{L})$, which have valuation equal to 1 in $\tilde{D}\det(\underline{L})$, and possibly 2. Lastly, primes that divide $\mathfrak{f}$, but not $\mathfrak{d}$, are those odd primes dividing $\det(\underline{L})$, which have positive, even valuation in $\tilde{D}\det(\underline{L})$, and possibly 2.

In the case where $\mathrm{rk}(\underline{L})$ is even, Definition 1.9 implies that

$$\Delta(\underline{L}) = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} \det(\underline{L}) = (-1)^{\frac{\mathrm{rk}(\underline{L})}{2}} 2^{v_2(\det(\underline{L}))} \prod_{\substack{p\mid\det(\underline{L})\\ p\neq 2}} p^{v_p(\det(\underline{L}))}.$$

It follows that we must distinguish between the cases where $\mathrm{rk}(\underline{L}) \equiv 0 \bmod 4$ and $\mathrm{rk}(\underline{L}) \equiv 2 \bmod 4$. For every odd prime $p \mid \det(\underline{L})$, define the constant

$$g_p := \begin{cases} 1, & \text{if } v_p(\det(\underline{L})) \equiv 1 \bmod 2 \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Let S denote the set of primes $\{p \equiv 3 \bmod 4 : p \mid \det(\underline{L}), g_p = 1)\}$. If $\mathrm{rk}(\underline{L}) \equiv 2 \bmod 4$, then $\Delta(\underline{L})$ is negative and therefore

$$\mathfrak{d} = -\prod_{\substack{p \mid \det(\underline{L}) \\ p \neq 2}} p^{g_p} \times \begin{cases} 1, & \text{if } v_2(\det(\underline{L})) \text{ is even and } |S| \text{ is odd,} \\ 4, & \text{if } v_2(\det(\underline{L})) \text{ and } |S| \text{ are even and} \\ 8, & \text{if } v_2(\det(\underline{L})) \text{ is odd} \end{cases}$$

and

$$\mathfrak{f} = \prod_{\substack{p \mid \det(\underline{L}) \\ p \neq 2}} p^{\lfloor \frac{v_p(\det(\underline{L}))}{2} \rfloor} \times \begin{cases} 2^{\frac{v_2(\det(\underline{L}))}{2}}, & \text{if } v_2(\det(\underline{L})) \text{ is even and } |S| \text{ is odd,} \\ 2^{\frac{v_2(\det(\underline{L})/4)}{2}}, & \text{if } v_2(\det(\underline{L})) \text{ and } |S| \text{ are even and} \\ 2^{\frac{v_2(\det(\underline{L})/8)}{2}}, & \text{if } v_2(\det(\underline{L})) \text{ is odd.} \end{cases}$$

On the other hand, if $\mathrm{rk}(\underline{L}) \equiv 0 \bmod 4$, then $\Delta(\underline{L})$ is positive and therefore

$$\mathfrak{d} = \prod_{\substack{p \mid \det(\underline{L}) \\ p \neq 2}} p^{g_p} \times \begin{cases} 1, & \text{if } v_2(\det(\underline{L})) \text{ and } |S| \text{ are even,} \\ 4, & \text{if } v_2(\det(\underline{L})) \text{ is even and } |S| \text{ is odd and} \\ 8, & \text{if } v_2(\det(\underline{L})) \text{ is odd} \end{cases}$$

and

$$\mathfrak{f} = \prod_{\substack{p \mid \det(\underline{L}) \\ p \neq 2}} p^{\lfloor \frac{v_p(\det(\underline{L}))}{2} \rfloor} \times \begin{cases} 2^{\frac{v_2(\det(\underline{L}))}{2}}, & \text{if } v_2(\det(\underline{L})) \text{ and } |S| \text{ are even,} \\ 2^{\frac{v_2(\det(\underline{L})/4)}{2}}, & \text{if } v_2(\det(\underline{L})) \text{ is even and } |S| \text{ is odd and} \\ 2^{\frac{v_2(\det(\underline{L})/8)}{2}}, & \text{if } v_2(\det(\underline{L})) \text{ is odd.} \end{cases}$$

It follows that, when $\mathrm{rk}(\underline{L})$ is even, the primes that divide both $\mathfrak{d}$ and $\mathfrak{f}$ are those odd primes dividing $\det(\underline{L})$, which have odd valuation greater than or equal to 3 in $\det(\underline{L})$, and possibly the prime 2. Primes that divide $\mathfrak{d}$, but not $\mathfrak{f}$, are those odd primes dividing $\det(\underline{L})$, which have valuation equal to 1 in $\det(\underline{L})$, and possibly 2. Lastly, primes that divide $\mathfrak{f}$, but not $\mathfrak{d}$, are those odd primes dividing $\det(\underline{L})$, which have positive, even valuation in $\det(\underline{L})$, and possibly 2.

Example 2.17 (Eisenstein-series of index $\underline{L}_1$)). The lattice $\underline{L}_1 = (\mathbb{Z}, (x, y) \mapsto 2xy$ has rank one and its dual is $\frac{1}{2}\mathbb{Z}$. It follows that its determinant is equal to 2 and its level equals its discriminant $\Delta(\underline{L}_1) = 4$. The discriminant module of $\underline{L}_1$ is $D_{\underline{L}_1} = \left(\left\{0, \frac{1}{2}\right\}, x \mapsto x^2 \bmod \mathbb{Z}\right)$ and hence $J^{\mathrm{Eis}}_{k,\underline{L}_1} = \mathbb{C}E_{k,\underline{L}_1,0}$ is a one dimensional vector space over $\mathbb{C}$. The theta series in the singular term of $E_{k,\underline{L}_1,0}$ is

$$\vartheta_{\underline{L}_1,0}(\tau, z) = \sum_{x \in \mathbb{Z}} e\left(x^2\tau + 2xz\right),$$

which is one of the four classical Jacobi theta functions. The Eisenstein series $E_{k,\underline{L}_1,0}$ has the following Fourier expansion:

$$\begin{aligned} E_{k,\underline{L}_1,0}(\tau, z) =& \vartheta_{\underline{L}_1,0}(\tau, z) + \sum_{\substack{x \in \frac{1}{2}\mathbb{Z}, D \in \mathbb{Q}_{<0} \\ D \equiv x^2 \bmod \mathbb{Z}}} G_{k,\underline{L}_1,0}(D, x) e\left(\left(x^2 - D\right)\tau + 2xz\right) \\ =& \vartheta_{\underline{L}_1,0}(\tau, z) + \sum_{x \in \mathbb{Z}, D \in \mathbb{Z}_{<0}} G_{k,\underline{L}_1,0}(D, x) e\left(\left(x^2 - D\right)\tau + 2xz\right) \\ &+ \sum_{\substack{x \in \mathbb{Z}+\frac{1}{2}, D \equiv \frac{1}{4} \bmod \mathbb{Z} \\ D<0}} G_{k,\underline{L}_1,0}(D, x) e\left(\left(x^2 - D\right)\tau + 2xz\right). \end{aligned}$$

The Fourier coefficients $G_{k,\underline{L}_1,0}(D,x)$ can be computed using (2.30). If $x \in \mathbb{Z}$, then $N_x = 1$. Write $D = D_0 f^2$, with

$$f = \prod_{p|D \text{ is odd}} p^{\lfloor \frac{v_p(D)}{2} \rfloor} \qquad \text{and} \qquad D_0 = -2^{v_2(D)} \prod_{\substack{p|D \text{ is odd} \\ v_p(D) \text{ is odd}}} p;$$

it follows that $\tilde{D} = D$ and $\tilde{D}_{0,x} = D_0$. The discriminant of $\mathbb{Q}\left(\sqrt{4D_0}\right)$ is

$$\mathfrak{d} := \mathfrak{d}_{D,x} = \begin{cases} \frac{D_0}{2^{v_2(D)}}, & \text{if } v_2(D) \text{ is even and } \#\{p \mid D_0 : p \equiv 3 \bmod 4\} \text{ is odd,} \\ \frac{4D_0}{2^{v_2(D)}}, & \text{if } v_2(D) \text{ and } \#\{p \mid D_0 : p \equiv 3 \bmod 4\} \text{ are even and} \\ \frac{8D_0}{2^{v_2(D)}}, & \text{if } v_2(D) \text{ is odd.} \end{cases}$$

and therefore

$$\mathfrak{f} := \mathfrak{f}_{D,x} = \begin{cases} 2^{\frac{v_2(D)+2}{2}}, & \text{if } v_2(D) \text{ is even and } \#\{p \mid D_0 : p \equiv 3 \bmod 4\} \text{ is odd,} \\ 2^{\frac{v_2(D)}{2}}, & \text{if } v_2(D) \text{ and } \#\{p \mid D_0 : p \equiv 3 \bmod 4\} \text{ are even and} \\ 2^{\frac{v_2(D)-1}{2}}, & \text{if } v_2(D) \text{ is odd.} \end{cases}$$

The bad primes dividing $\det(\underline{L}_1)\tilde{D}$ are 2 and the odd primes dividing D. The quadratic character $\chi_{\underline{L}_1}(\tilde{D}_0, \cdot)$ is equal to $\left(\frac{4D_0}{\cdot}\right)$ and therefore $\chi_{\underline{L}_1}(\tilde{D}_0, 2) = 0$ and $\chi_{\underline{L}_1}(\tilde{D}_0, p) = \chi_{\mathfrak{d}}(p)$ for all odd primes. Thus, when $x \in \mathbb{Z}$, we obtain that

$$G_{k,\underline{L}_1,0}(D,x) = -\frac{2^{2k-1}(k-1)(DD_0)^{\frac{1}{2}}D^{k-2}}{B_{2k-2}\mathfrak{f}|\mathfrak{d}|^{k-1}} L_{\mathfrak{d}}(2-k) \times \frac{\tilde{L}_2(k-1)}{1-2^{2-2k}} \\ \times \sum_{d|\mathfrak{f}} \mu(d)\left(\frac{\mathfrak{d}}{d}\right) d^{1-k}\sigma_{3-2k}\left(\frac{\mathfrak{f}}{d}\right) \prod_{p|D, p\neq 2} \frac{1-\chi_{\mathfrak{d}}(p)p^{1-k}}{1-p^{2-2k}} \tilde{L}_p(k-1).$$

Lemma 2.16 implies that

$$\sum_{d|\mathfrak{f}} \mu(d)\left(\frac{\mathfrak{d}}{d}\right) d^{1-k}\sigma_{3-2k}\left(\frac{\mathfrak{f}}{d}\right) = 1 + \left(1-\chi_{\mathfrak{d}}(2)2^{k-2}\right)\frac{1-2^{v_2(\mathfrak{f})(3-2k)}}{2^{2k-3}-1}.$$

Combining this with (2.34), we obtain that

$$G_{k,\underline{L}_1,0}(D,x) = \frac{f\mathfrak{f}D^{k-2}}{B_{2k-2}} \sum_{m=1}^{|\mathfrak{d}|} \chi_{\mathfrak{d}}(m) B_{k-1}\left(\frac{m}{|\mathfrak{d}|}\right) \times \frac{2^{2k-3}\tilde{L}_2(k-1)}{1-2^{2-2k}} \\ \times \left(1 + \left(1-\chi_{\mathfrak{d}}(2)2^{k-2}\right)\frac{1-2^{v_2(\mathfrak{f})(3-2k)}}{2^{2k-3}-1}\right) \prod_{p|D \text{ is odd}} \frac{1-\chi_{\mathfrak{d}}(p)p^{1-k}}{1-p^{2-2k}} \tilde{L}_p(k-1).$$

When $x \in \mathbb{Z} + \frac{1}{2}$, its order is equal to 2 and $D \in \frac{1}{4}(4\mathbb{Z}+1)$. We have $D = D_0 f^2$, with

$$f = \prod_{p|4D} p^{\lfloor \frac{v_p(4D)}{2} \rfloor} \qquad \text{and} \qquad D_0 = -\frac{1}{4} \prod_{\substack{p|4D \\ v_p(4D) \text{ is odd}}} p;$$

it follows that $\tilde{D} = 4D$ and $\tilde{D}_{0,x} = 4D_0$. The discriminant of $\mathbb{Q}\left(\sqrt{4\tilde{D}_0}\right)$ is

$$\mathfrak{d} := \mathfrak{d}_{D,x} = \tilde{D}_{0,x} = -\prod_{\substack{p|4D \\ v_p(4D) \text{ is odd}}} p$$

and therefore $\mathfrak{f}_{D,x} = 2$. This is because $4D_0 \equiv 4D \equiv 1 \bmod 4$. The bad primes dividing $\det(\underline{L}_1)\tilde{D}$ are 2 and the primes dividing $4D$. The quadratic character $\chi_{\underline{L}_1}(\tilde{D}_0, \cdot)$ is equal to

$\left(\frac{16D_0}{\cdot}\right)$ and therefore $\chi_{\underline{L}_1}(\tilde{D}_0, 2) = 0$ and $\chi_{\underline{L}_1}(\tilde{D}_0, p) = \chi_{4D_0}(p)$ for all odd primes. Thus, when $x \in \mathbb{Z} + \frac{1}{2}$, we obtain that

$$\begin{aligned}
G_{k,\underline{L}_1,0}(D, x) = &- \frac{2^{2k-2}\,(k-1)\,(4DD_0)^{\frac{1}{2}}\,D^{k-2}}{B_{2k-2}|4D_0|^{k-1}} L_{4D_0}\,(2-k)\,\frac{\tilde{L}_2(k-1)}{1-2^{2-2k}} \\
&\times \left(1 + 2^{3-2k} - \chi_{4D_0}(2)2^{1-k}\right) \prod_{p|4D} \frac{1-\chi_{4D_0}(p)p^{1-k}}{1-p^{2-2k}} \tilde{L}_p(k-1) \\
=&\frac{fD^{k-2}}{B_{2k-2}} \sum_{m=1}^{|4D_0|} \chi_{4D_0}(m) B_{k-1}\left(\frac{m}{|4D_0|}\right)\left(1 + 2^{2k-3} - \chi_{4D_0}(2)2^{k-2}\right) \\
&\times \frac{\tilde{L}_2(k-1)}{1-2^{2-2k}} \prod_{p|4D} \frac{1-\chi_{4D_0}(p)p^{1-k}}{1-p^{2-2k}} \tilde{L}_p(k-1).
\end{aligned}$$

It follows that

$$\begin{aligned}
E_{k,\underline{L}_1,0}(\tau, z) =&\vartheta_{\underline{L}_1,0}(\tau, z) + \sum_{x\in\mathbb{Z}, D\in\mathbb{Z}_{<0}} \frac{f\mathfrak{f}D^{k-2}}{B_{2k-2}}\Bigg[\sum_{m=1}^{|\mathfrak{d}|}\chi_{\mathfrak{d}}(m)B_{k-1}\left(\frac{m}{|\mathfrak{d}|}\right)\frac{\tilde{L}_2(k-1)}{1-2^{2-2k}} \\
&\times\left(2^{2k-3} + \left(1-\chi_{\mathfrak{d}}(2)2^{k-2}\right)\frac{1-2^{\nu_2(\mathfrak{f})(3-2k)}}{1-2^{3-2k}}\right) \\
&\times \prod_{p|D, p\neq 2} \frac{1-\chi_{\mathfrak{d}}(p)p^{1-k}}{1-p^{2-2k}}\tilde{L}_p(k-1)\Bigg] e\left(\left(x^2 - D\right)\tau + 2xz\right) \\
&+ \sum_{\substack{x\in\mathbb{Z}+\frac{1}{2}, D\equiv\frac{1}{4} \bmod \mathbb{Z} \\ D<0}} \frac{fD^{k-2}}{B_{2k-2}}\Bigg[\sum_{m=1}^{|4D_0|}\chi_{4D_0}(m)B_{k-1}\left(\frac{m}{|4D_0|}\right)\left(2^{2k-3} + 1 - \chi_{4D_0}(2)2^{k-2}\right) \\
&\times \frac{\tilde{L}_2(k-1)}{1-2^{2-2k}} \prod_{p|4D}\frac{1-\chi_{4D_0}(p)p^{1-k}}{1-p^{2-2k}}\tilde{L}_p(k-1)\Bigg] e\left(\left(x^2-D\right)\tau + 2xz\right).
\end{aligned}$$

In the following subsection, we discuss an alternative method for computing the Euler factors $\tilde{L}_p$, which also works for the bad primes $p \mid 2\tilde{D}\det(\underline{L})$.

2.3.1. Igusa zeta functions and representation numbers. A different method to compute the Euler factors (2.25) is based on results from [**CKW17**] on calculating the Igusa local zeta function (see Definition 1.1). We remind the reader of the definition (2.18) of the representation numbers $R_b(Q)$. For every quadratic polynomial f in $\mathbb{Z}_p[X_1, \ldots, X_{\mathrm{rk}(\underline{L})}]$, the following holds:

$$\frac{1 - p^{-s}\zeta(f; p; s)}{1-p^{-s}} = \sum_{l=0}^{\infty} R_{p^l}(f)p^{-l(s+\mathrm{rk}(\underline{L}))},$$

in other words

$$L_p(s) = \frac{p^{s-\mathrm{rk}(\underline{L})} - \zeta(Q_{D,x}; p; s-\mathrm{rk}(\underline{L}))}{p^{s-\mathrm{rk}(\underline{L})} - 1}. \tag{2.36}$$

A quadratic form $Q : \mathbb{Z}_p^n \to \mathbb{Z}_p$ is said to be *unimodular* if the determinant of its Gram matrix is invertible in $\mathbb{Z}_p$. By the direct sum of two quadratic forms $Q_1 : \mathbb{Z}_p^n \to \mathbb{Z}_p$ and $Q_2 : \mathbb{Z}_p^m \to \mathbb{Z}_p$ we mean the quadratic form $Q_1 \oplus Q_2 : \mathbb{Z}_p^{n+m} \to \mathbb{Z}_p$,

$$Q_1 \oplus Q_2(x_1, \ldots, x_n, y_1, \ldots, y_m) = Q_1(x_1, \ldots, x_n) + Q_2(y_1, \ldots, y_m).$$

A quadratic polynomial Q is said to be in *normal form* if $Q = \oplus_{i\in\mathbb{N}\cup\{0\}} p^i Q_i + c$, where each Q_i is a unimodular quadratic form over $\mathbb{Z}_p$ and c is a constant in $\mathbb{Z}_p$. It was shown in [**CKW17**, §4.9] that every quadratic polynomial f as above is *isospectral at p* to a polynomial Q which is in normal form, meaning that $R_{p^l}(f) = R_{p^l}(Q)$ for ever l in $\mathbb{N}$.

For every prime p and every integers a, r and d, define the following helper function:

- if r is odd, then

$$I_a(r,d)(s) := \begin{cases} (1-p^{-s-r})\dfrac{p-1}{p-p^{-s}}, & p \mid a, \\ \left[1+p^{-s-\lceil\frac{r}{2}\rceil}\left(\frac{ad(-1)^{\lceil\frac{r}{2}\rceil}}{p}\right)\right]\dfrac{p-1}{p-p^{-s}} - \dfrac{1}{p^r} - \dfrac{1}{p^{\lceil\frac{r}{2}\rceil}}\left(\frac{ad(-1)^{\lceil\frac{r}{2}\rceil}}{p}\right), & p \nmid a; \end{cases}$$

- if r is even, then

$$I_a(r,d)(s) := \begin{cases} \left[1-p^{-\frac{r}{2}}\left(\frac{(-1)^{-\frac{r}{2}}d}{p}\right)\right]\left[1+p^{-s-\frac{r}{2}}\left(\frac{(-1)^{-\frac{r}{2}}d}{p}\right)\right]\dfrac{p-1}{p-p^{-s}}, & p \mid a, \\ \left[1-p^{-\frac{r}{2}}\left(\frac{(-1)^{-\frac{r}{2}}d}{p}\right)\right]\left[\dfrac{p-1}{p-p^{-s}} + p^{-\frac{r}{2}}\left(\frac{(-1)^{-\frac{r}{2}}d}{p}\right)\right], & p \nmid a. \end{cases}$$

THEOREM 2.18 ([**CKW17**, Thm 2.1]). *Let p be an odd prime and $Q = \oplus_{i\in\mathbb{N}\cup\{0\}} p^i Q_i + c$ be a quadratic polynomial over $\mathbb{Z}_p$ which is in normal form, where each Q_i is a unimodular quadratic form of rank r_i and discriminant d_i over $\mathbb{Z}_p$ and c is a constant in $\mathbb{Z}_p$. Set $\kappa := v_p(c)$ and, for every j in $\mathbb{N}\cup\{0\}$, set*

$$Q(j) := \bigoplus_{\substack{0\le i\le j \\ i\equiv j \bmod 2}} Q_i, \qquad d(j) := disc(Q(j)) = \prod_{\substack{0\le i\le j \\ i\equiv j \bmod 2}} d_i,$$

$$r(j) := rk(Q(j)) = \sum_{\substack{0\le i\le j \\ i\equiv j \bmod 2}} r_i, \qquad p(j) := p^{\sum_{0\le i<j} r(i)}.$$

Then

$$\zeta(Q;p;s) = \sum_{0\le l\le\kappa} \frac{I_{c/p^l}\left(r(l),d(l)\right)(s)}{p(l)} p^{-ls} + \frac{1}{p(\kappa+1)} p^{-\kappa s}.$$

When $p = 2$, the Igusa zeta function $\zeta(2;s)$ can be computed using [**CKW17**, Theorem 2.3].

Set $\zeta(p;s) := \zeta(Q_{D,x};p;s)$ for simplicity. Let us redo the calculations for $L_p(s)$ at good primes p using (2.36) and Theorem 2.18. We remind the reader that, when $(p, 2\det(\underline{L})\tilde{D}) = 1$, (2.23) implies that

$$R_{p^l} = \#\{\lambda \in L/p^l L : 2\beta(\lambda) - 2\tilde{D} \equiv 0 \bmod p^l\} = R_{p^l}(2\beta(\lambda) - 2\tilde{D}), \tag{2.37}$$

and the quadratic polynomial $2\beta(\lambda) - 2\tilde{D}$ is in normal form. Since $\kappa = 0$ in this case, it follows that

$$\zeta(p;s) = \frac{I_{-2\tilde{D}}\left(r(0),d(0)\right)(s)}{p(0)} + \frac{1}{p(1)}.$$

We have

$$d(0) = d_0 = \det(\underline{L}), \qquad r(0) = r_0 = \mathrm{rk}(\underline{L}),$$
$$p(0) = p^0 = 1, \qquad p(1) = p^{r(0)} = p^{\mathrm{rk}(\underline{L})}.$$

If $\mathrm{rk}(\underline{L})$ is even, then

$$I_{-2\tilde{D}}\left(r(0),d(0)\right)(s) = \left(1 - p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p)\right)\left(\frac{p-1}{p-p^{-s}} + p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p)\right)$$

and Theorem 2.18 implies that

$$\zeta(p;s) = \frac{1 - p - p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p) + p^{-\frac{\mathrm{rk}(\underline{L})}{2}-s}\chi_{\underline{L}}(p)}{p^{-s}-p}$$

after a straight-forward calculation. It follows from (2.36) that

$$\begin{aligned}
L_p(s) =& \frac{p^{-(s-\mathrm{rk}(\underline{L}))}\left(p + p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p) - p^{-\frac{\mathrm{rk}(\underline{L})}{2}-(s-\mathrm{rk}(\underline{L}))}\chi_{\underline{L}}(p)\right) - p}{(p^{-(s-\mathrm{rk}(\underline{L}))} - p)(1 - p^{-(s-\mathrm{rk}(\underline{L}))})} \\
=& \frac{p^{-(s-\mathrm{rk}(\underline{L}))} - 1 + p^{-s+\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p)\left(1 - p^{-(s-\mathrm{rk}(\underline{L}))}\right)}{(p^{-(s-\mathrm{rk}(\underline{L})+1)} - 1)(1 - p^{-(s-\mathrm{rk}(\underline{L}))})} \\
=& \frac{1 - \chi_{\underline{L}}(p)p^{-\left(s-\frac{rk}{2}+1\right)}}{1 - p^{-(s-\mathrm{rk}(\underline{L})+1)}}
\end{aligned}$$

and these calculations lead to the same result as Lemma 2.13.

If $\mathrm{rk}(\underline{L})$ is odd, then $-2\det(\underline{L})(-1)^{\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil} = \Delta(\underline{L})$ and $\tilde{D}_0 f^2 = \tilde{D}$ and hence

$$\left(\frac{-2\tilde{D}(-1)^{\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}\det(\underline{L})}{p}\right) = \chi_{\underline{L}}(\tilde{D}_0, p).$$

Thus,

$$I_{-2\tilde{D}}\left(r(0), d(0)\right)(s) = \left(1 + p^{-s-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}\chi_{\underline{L}}(\tilde{D}_0, p)\right)\frac{p-1}{p-p^{-s}} - p^{-\mathrm{rk}(\underline{L})} - \frac{\chi_{\underline{L}}(\tilde{D}_0, p)}{p^{\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}}$$

and hence

$$\zeta(p;s) = \frac{p - 1 + \chi_{\underline{L}}(\tilde{D}_0, p)p^{1-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}(p^{-s} - 1)}{p - p^{-s}}$$

after a straight-forward calculation. It follows that

$$\begin{aligned}
L_p(s) =& \frac{p - p^{-(s-\mathrm{rk}(\underline{L}))}\left(p + \chi_{\underline{L}}(\tilde{D}_0, p)p^{1-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil}\left(p^{-(s-\mathrm{rk}(\underline{L}))} - 1\right)\right)}{(p - p^{-(s-\mathrm{rk}(\underline{L}))})(1 - p^{-(s-\mathrm{rk}(\underline{L}))})} \\
=& \frac{\left(1 - p^{-(s-\mathrm{rk}(\underline{L}))}\right) + \chi_{\underline{L}}(\tilde{D}_0, p)p^{-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-s+\mathrm{rk}(\underline{L})}\left(1 - p^{-(s-\mathrm{rk}(\underline{L}))}\right)}{(1 - p^{-(s-\mathrm{rk}(\underline{L})+1)})(1 - p^{-(s-\mathrm{rk}(\underline{L}))})} \\
=& \frac{1 + \chi_{\underline{L}}(\tilde{D}_0, p)p^{-\left(s-\lfloor\frac{rk}{2}\rfloor\right)}}{1 - p^{-(s-\mathrm{rk}(\underline{L})+1)}}
\end{aligned}$$

and these calculations lead to the same result as Lemma 2.13.

If p is a prime such that $(p, 2\det(\underline{L})) = 1$, but $v_p(\tilde{D}) > 0$, then (2.37) still holds and Theorem 2.18 can be used to compute $L_p(\cdot)$. We illustrate this method in the case where $\mathrm{rk}(\underline{L})$ is even.

Proposition 2.19. *If* $\mathrm{rk}(\underline{L})$ *is even and* p *is a prime such that* $(p, 2\det(\underline{L})) = 1$ *and* $\kappa := v_p(\tilde{D}) > 0$*, then the following holds:*

$$\tilde{L}_p(s) = \chi_{\underline{L}}(p^\kappa)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\left(1 - \chi_{\underline{L}}(p)p^{-\left(s-\frac{\mathrm{rk}(\underline{L})}{2}+1\right)}\right)\frac{\left(\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}\right)^{\kappa+1} - 1}{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}} - 1}.$$

Furthermore, set $\mathfrak{g} := \prod_{p\mid\tilde{D}, p\nmid 2\det(\underline{L})} p^{\nu_p(\tilde{D})}$. *Then*

$$\prod_{\substack{p\mid\tilde{D}\\ p\nmid 2\det(\underline{L})}} \frac{\tilde{L}_p(k-1)}{1-\chi_{\mathfrak{d}}(p)p^{-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}} = \chi_{\mathfrak{d}}(\mathfrak{g})\mathfrak{g}^{-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}-1\right)}\sigma^{\chi_{\mathfrak{d}}}_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}(\mathfrak{g}).$$

Proof. Consider $Q = 2\beta - 2\tilde{D}$ in Theorem 2.18. Then

$$d(l) = \begin{cases}\det(\underline{L}), & l \equiv 0 \bmod 2,\\ 1, & l \equiv 1 \bmod 2,\end{cases} \qquad r(l) = \begin{cases}\mathrm{rk}(\underline{L}), & l \equiv 0 \bmod 2,\\ 0, & l \equiv 1 \bmod 2,\end{cases}$$

$$p(l) = p^{\mathrm{rk}(\underline{L})\lceil \frac{l}{2}\rceil}$$

for every positive integer l.

Suppose that κ is odd. Then

$$I_{-2\tilde{D}/p^{\kappa}}\left(r(\kappa), d(\kappa)\right)(s) = I_{-2\tilde{D}/p^{\kappa}}\left(0,1\right)(s) = 0.$$

For every $0 \le l < \kappa$, we have $p \mid (-2\tilde{D}/p^l)$ and hence, if l is odd, then

$$I_{-2\tilde{D}/p^{l}}\left(r(l), d(l)\right)(s) = I_{-2\tilde{D}/p^{l}}\left(0,1\right)(s) = 0 \tag{2.38}$$

and, if l is even, then

$$\begin{aligned} I_{-2\tilde{D}/p^{l}}\left(r(l), d(l)\right)(s) &= I_{-2\tilde{D}/p^{l}}(\mathrm{rk}(\underline{L}), \det(\underline{L}))(s)\\ &= \left[1-p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p)\right]\left[1+p^{-s-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p)\right]\frac{p-1}{p-p^{-s}}. \end{aligned} \tag{2.39}$$

Theorem 2.18 implies that

$$\begin{aligned} \zeta(p;s) = &\sum_{0\le l\le \frac{\kappa-1}{2}} I_{-2\tilde{D}/p^{2l}}\left(\mathrm{rk}(\underline{L}), \det(\underline{L})\right)(s) p^{-l(\mathrm{rk}(\underline{L})+2s)}\\ &+ p^{-s-\mathrm{rk}(\underline{L})}\sum_{0\le l\le \frac{\kappa-1}{2}} I_{-2\tilde{D}/p^{2l+1}}\left(0,1\right)(s)p^{-l(\mathrm{rk}(\underline{L})+2s)} + p^{-\kappa\left(s+\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}}\\ = &\left[1-\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right]\left[1+\chi_{\underline{L}}(p)p^{-s-\frac{\mathrm{rk}(\underline{L})}{2}}\right]\frac{p-1}{p-p^{-s}}\sum_{0\le l\le \frac{\kappa-1}{2}} p^{-l\left(\mathrm{rk}(\underline{L})+2s\right)}\\ &+ p^{-\kappa\left(s+\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}}. \end{aligned}$$

The geometric progression sum in the above equation can be computed using the formula

$$\sum_{0\le l\le T} p^{-lt} = \frac{p^t - p^{-Tt}}{p^t-1}. \tag{2.40}$$

In order to apply (2.36), we need to compute $\zeta(p; s - \mathrm{rk}(\underline{L}))$. We have

$$p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}} = \frac{p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}} - p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}} - p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1} + p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}}{\left(1-p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)\left(p^{2s-\mathrm{rk}(\underline{L})}-1\right)}$$

and

$$\Big[1-\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\Big]\Big[1+\chi_{\underline{L}}(p)p^{-s+\frac{\mathrm{rk}(\underline{L})}{2}}\Big]\frac{p-1}{p-p^{-s+\mathrm{rk}(\underline{L})}}\sum_{0\le l\le\frac{\kappa-1}{2}}p^{-l\left(2s-\mathrm{rk}(\underline{L})\right)}$$
$$\begin{aligned}=&\left(1-p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)^{-1}\left(p^{2s-\mathrm{rk}(\underline{L})}-1\right)^{-1}\Big[p^{2s-\mathrm{rk}(\underline{L})}-p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-p^{2s-\mathrm{rk}(\underline{L})-1}\\
&+p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}+\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-\chi_{\underline{L}}(p)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}-1}\\
&+\chi_{\underline{L}}(p)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}-\chi_{\underline{L}}(p)p^{2s-\frac{3\mathrm{rk}(\underline{L})}{2}}+\chi_{\underline{L}}(p)p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}}+\chi_{\underline{L}}(p)p^{2s-\frac{3\mathrm{rk}(\underline{L})}{2}-1}\\
&-\chi_{\underline{L}}(p)p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}-1}-p^{s-\mathrm{rk}(\underline{L})}+p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}}+p^{s-\mathrm{rk}(\underline{L})-1}-p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}-1}\Big].\end{aligned}$$

Note that

$$p^{s-\mathrm{rk}(\underline{L})}=\frac{p^{3s-2\mathrm{rk}(\underline{L})}-p^{2s-\mathrm{rk}(\underline{L})-1}-p^{s-\mathrm{rk}(\underline{L})}+p^{-1}}{\left(1-p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)\left(p^{2s-\mathrm{rk}(\underline{L})}-1\right)}. \tag{2.41}$$

The above calculations combined with (2.36) imply that

$$\begin{aligned}L_p(s)=&\Big(p^{2s-\mathrm{rk}(\underline{L})}-p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}+\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-\chi_{\underline{L}}(p)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}-1}-p^{-1}\\
&+\chi_{\underline{L}}(p)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}+p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}\Big)\left(1-p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)^{-1}\left(p^{2s-\mathrm{rk}(\underline{L})}-1\right)^{-1}.\end{aligned}$$

It follows from (2.26) and from writing

$$p^{2s-\mathrm{rk}(\underline{L})}-1=\left(\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-1\right)\left(\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}+1\right) \tag{2.42}$$

that

$$\begin{aligned}\tilde{L}_p(s)=&\frac{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-\chi_{\underline{L}}(p)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-p^{-1}+p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}}{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-1}\\
=&\frac{\left(\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-p^{-1}\right)\left(1-p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\right)}{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-1}\\
=&\chi_{\underline{L}}(p^\kappa)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\left(1-\chi_{\underline{L}}(p)p^{-\left(s-\frac{\mathrm{rk}(\underline{L})}{2}+1\right)}\right)\frac{\left(\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}\right)^{(\kappa+1)}-1}{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-1},\end{aligned}$$

as claimed. We have used the fact that $\chi_{\underline{L}}(p)^{\kappa+1}=\chi_{\underline{L}}(p)^{\kappa-1}=1$ when κ is odd.

Next, suppose that κ is even. Then

$$\begin{aligned}I_{-2\tilde{D}/p^\kappa}\left(r(\kappa),d(\kappa)\right)(s)=&I_{-2\tilde{D}/p^\kappa}\left(\mathrm{rk}(\underline{L}),\det(\underline{L})\right)(s)\\
=&\Big[1-p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p)\Big]\Big[\frac{p-1}{p-p^{-s}}+p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\chi_{\underline{L}}(p)\Big].\end{aligned}$$

Theorem 2.18 and equations (2.38) and (2.39) imply that

$$\begin{aligned}\zeta(p;s) &= \sum_{0\le l\le \frac{\kappa}{2}} I_{-2\tilde{D}/p^{2l}}\left(\mathrm{rk}(\underline{L}),\det(\underline{L})\right)(s)p^{-l(\mathrm{rk}(\underline{L})+2s)}\\ &\quad + p^{-s-\mathrm{rk}(\underline{L})}\sum_{0\le l\le\frac{\kappa-2}{2}} I_{-2\tilde{D}/p^{2l+1}}\left(0,1\right)(s)p^{-l(\mathrm{rk}(\underline{L})+2s)} + p^{-\kappa\left(s+\frac{\mathrm{rk}(\underline{L})}{2}\right)-\mathrm{rk}(\underline{L})}\\ &=\left[1-\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right]\left[1+\chi_{\underline{L}}(p)p^{-s-\frac{\mathrm{rk}(\underline{L})}{2}}\right]\frac{p-1}{p-p^{-s}}\sum_{0\le l\le\frac{\kappa-2}{2}} p^{-l\left(\mathrm{rk}(\underline{L})+2s\right)}\\ &\quad+\left[\left(1-\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right)\frac{p-1}{p-p^{-s}}+\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right]p^{-\kappa\left(s+\frac{\mathrm{rk}(\underline{L})}{2}\right)},\end{aligned}$$

since $I_{-2\tilde{D}/p^{2l+1}}\left(0,1\right)(s)$ vanishes for every l in the second sum. We need to compute $\zeta(p;s-\mathrm{rk}(\underline{L}))$. We have

$$\begin{aligned}&\left[\left(1-\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right)\frac{p-1}{p-p^{-s+\mathrm{rk}(\underline{L})}}+\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right]p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\\ &=\left(1-p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)^{-1}\left(p^{2s-\mathrm{rk}(\underline{L})}-1\right)^{-1}\Big(p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}-p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\\ &\quad+p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}+\chi_{\underline{L}}(p)p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}-1}-\chi_{\underline{L}}(p)p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}\\ &\quad-\chi_{\underline{L}}(p)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}-1}+\chi_{\underline{L}}(p)p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}\Big)\end{aligned}$$

and, using (2.40),

$$\begin{aligned}&\left[1-\chi_{\underline{L}}(p)p^{-\frac{\mathrm{rk}(\underline{L})}{2}}\right]\left[1+\chi_{\underline{L}}(p)p^{-s+\frac{\mathrm{rk}(\underline{L})}{2}}\right]\frac{p-1}{p-p^{-s+\mathrm{rk}(\underline{L})}}\sum_{0\le l\le\frac{\kappa-2}{2}}p^{-l\left(2s-\mathrm{rk}(\underline{L})\right)}\\ &=\left(1-p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)^{-1}\left(p^{2s-\mathrm{rk}(\underline{L})}-1\right)^{-1}\Big[p^{2s-\mathrm{rk}(\underline{L})}-p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-p^{2s-\mathrm{rk}(\underline{L})-1}\\ &\quad+p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}+\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-\chi_{\underline{L}}(p)p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}-1}\\ &\quad+\chi_{\underline{L}}(p)p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}-\chi_{\underline{L}}(p)p^{2s-\frac{3\mathrm{rk}(\underline{L})}{2}}+\chi_{\underline{L}}(p)p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}}\\ &\quad+\chi_{\underline{L}}(p)p^{2s-\frac{3\mathrm{rk}(\underline{L})}{2}-1}-\chi_{\underline{L}}(p)p^{-(\kappa-2)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}-1}-p^{s-\mathrm{rk}(\underline{L})}\\ &\quad+p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}}+p^{s-\mathrm{rk}(\underline{L})-1}-p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-\frac{\mathrm{rk}(\underline{L})}{2}-1}\Big].\end{aligned}$$

Equation (2.36) and the above calculations imply that

$$\begin{aligned}L_p(s)=&\Big(p^{2s-\mathrm{rk}(\underline{L})}+\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}-\chi_{\underline{L}}(p)p^{-(\kappa-1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}-\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}-1}-p^{-1}\\ &+p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}+\chi_{\underline{L}}(p)p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}\Big)\left(1-p^{-(s-\mathrm{rk}(\underline{L})+1)}\right)^{-1}\left(p^{2s-\mathrm{rk}(\underline{L})}-1\right)^{-1}.\end{aligned}$$

It follows from (2.26) and (2.42) that

$$\begin{aligned}
\tilde{L}_p(s) =& \frac{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}} - p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)} - p^{-1} + \chi_{\underline{L}}(p)p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)-1}}{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}} - 1}\\
=& \frac{\left(\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}} - p^{-1}\right)\left(1 - \chi_{\underline{L}}(p)p^{-(\kappa+1)\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\right)}{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}} - 1}\\
=& \chi_{\underline{L}}(p^\kappa)p^{-\kappa\left(s-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\left(1 - \chi_{\underline{L}}(p)p^{-\left(s-\frac{\mathrm{rk}(\underline{L})}{2}+1\right)}\right)\frac{\left(\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}}\right)^{(\kappa+1)} - 1}{\chi_{\underline{L}}(p)p^{s-\frac{\mathrm{rk}(\underline{L})}{2}} - 1}.
\end{aligned}$$

We have used the fact that $\chi_{\underline{L}}(p)^{\kappa+2} = \chi_{\underline{L}}(p)^{\kappa-2} = 1$ when κ is even.

We remind the reader that $\chi_{\underline{L}}(p) = \chi_{\mathfrak{d}}(p)$ for those primes which do not divide $\Delta(\underline{L})$. Hence, if we set $\mathfrak{g} := \prod_{p\mid\tilde{D}, p\nmid 2\det(\underline{L})} p^{\nu_p(\tilde{D})}$, then

$$\prod_{\substack{p\mid\tilde{D}\\ p\nmid 2\det(\underline{L})}} \frac{\tilde{L}_p(k-1)}{1-\chi_{\mathfrak{d}}(p)p^{-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}} = \chi_{\mathfrak{d}}(\mathfrak{g})\mathfrak{g}^{-\left(k-\frac{\mathrm{rk}(\underline{L})}{2}-1\right)}\sum_{d\mid\mathfrak{g}}\chi_{\mathfrak{d}}(d)d^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1},$$

as claimed. □

Example 2.20. If $\underline{L}$ is unimodular, then the only bad primes arising in (2.29) are $p = 2$ and the primes considered in Proposition 2.19. In this case, we have $\mathrm{rk}(\underline{L}) \equiv 0 \bmod 8$ and $\det(\underline{L}) = 1$. The latter implies that $N_x = 1$, that $\mathfrak{d} = \mathfrak{f} = 1$, that $\tilde{D} = D$ and that $\chi_{\underline{L}}(\cdot)$ is the trivial character. Combining this with (2.34), we obtain that

$$G_{k,\underline{L},0}(D,x) = -\frac{(2k-\mathrm{rk}(\underline{L}))\sigma_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}(-D)}{B_{k-\frac{\mathrm{rk}(\underline{L})}{2}}} \times \frac{\tilde{L}_2(k-1)2^{\nu_2(D)\left(k-\frac{\mathrm{rk}(\underline{L})}{2}-1\right)}\left(2^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}-1\right)}{\left(1-2^{\frac{\mathrm{rk}(\underline{L})}{2}-k}\right)\left(2^{(\nu_2(D)+1)\left(k-\frac{\mathrm{rk}(\underline{L})}{2}-1\right)}-1\right)}.$$

2.4. Fourier coefficients of non-trivial Eisenstein series

In this section, we use the notions discussed in Subsection 1.3.2 in order to obtain non-trivial linear relations between the Fourier coefficients of non-trivial Eisenstein series and those of the trivial one. As an application of this result, we obtain formulas for the Fourier coefficients of Eisenstein series associated with isotropic elements which have small order in the discriminant module of the lattice in the index.

Let φ denote the isomorphism between Jacobi forms and vector-valued modular forms from Theorem 1.39 and let σ_x denote the Schrödinger representation twisted at x from Definition 1.40. Define an *averaging operator* on $J_{k,\underline{L}}$ in the following way:

Definition 2.21. For every x in $L^\#/L$ and every ϕ in $J_{k,\underline{L}}$, set

$$\mathrm{Av}_x\,\phi(\tau,z) := \frac{1}{N_x^2}\sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2}\varphi^{-1}\sigma_x^*(m,n,0)\varphi(\phi)(\tau,z).$$

Remark 2.22. This operator was defined for vector-valued modular forms in [**Wil18**, §11]. The action of the Schrödinger representation (and implicitly Av_x) can be defined directly on theta series. However, we continue to work with vector-valued modular forms, since it is easier to prove modularity in this context.

The following holds:

Lemma 2.23. *The operator* Av_x *is well-defined (i.e. it does not depend on the choice of representatives of* $\mathbb{Z}_{(N_x^2)}$*) and it maps* $J_{k,\underline{L}}$ *to* $J_{k,\underline{L}}$.

Proof. For all integers u and v, we have

$$\sigma_x^*(m+uN_x^2,n,0)\mathfrak{e}_y = \sigma_x^*(m,n+vN_x^2,0)\mathfrak{e}_y = \sigma_x^*(m,n,0)\mathfrak{e}_y$$

and therefore Av_x is well-defined. To show that $\mathrm{Av}_x\,\phi$ is an element of $J_{k,\underline{L}}$, it suffices to prove that $\varphi\,(\mathrm{Av}_x\,\phi)$ is an element of $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\rho_{\underline{L}}^*)$. Set $F(\tau) := \varphi(\phi)$. For every pair (m,n) in $(\mathbb{Z}_{(N_x^2)})^2$ and every $\tilde{A}$ in $\tilde{\Gamma}$, the following holds:

$$\begin{aligned}(\sigma_x^*(m,n,0)F)\,|_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\tilde{A}(\tau) =& w(\tau)^{-2\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\sigma_x^*(m,n,0)F(A\tau)\\ =& w(\tau)^{-2\left(k-\frac{\mathrm{rk}(\underline{L})}{2}\right)}\rho_{\underline{L}}^*(\tilde{A})\sigma_x^*((m,n,0)^A)\rho_{\underline{L}}^*(\tilde{A})^{-1}F(A\tau)\\ =& \rho_{\underline{L}}^*(\tilde{A})\sigma_x^*((m,n,0)^A)F(\tau),\end{aligned}$$

using (1.27) in the middle line. Thus,

$$\begin{aligned}\varphi(\mathrm{Av}_x\,\phi)|_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\tilde{A}(\tau) =& \frac{1}{N_x^2}\sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2}\rho_{\underline{L}}^*(\tilde{A})\sigma_x^*((m,n,0)^A)F(\tau)\\ =& \frac{1}{N_x^2}\sum_{(m',n')\in(\mathbb{Z}_{(N_x^2)})^2}\rho_{\underline{L}}^*(\tilde{A})\sigma_x^*(m',n',0)F(\tau)\\ =& \rho_{\underline{L}}^*(\tilde{A})\varphi(\mathrm{Av}_x(\phi))(\tau),\end{aligned} \tag{2.43}$$

with the change of variable $(m',\mu') = ((m,n)A)$ (which is an isomorphism of $(\mathbb{Z}/N_x^2\mathbb{Z})^2$, since $A\in\Gamma$). Since clearly $\varphi(\mathrm{Av}_x(\phi))$ is holomorphic, it follows that it is an element of $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\rho_{\underline{L}}^*)$ and applying φ^{-1} to it completes the proof. □

Proposition 2.24. *For every* x *in* $L^\#/L$*, we have*

$$\mathrm{Av}_x\,E_{k,\underline{L},0}(\tau,z) = \sum_{\substack{m\in\mathbb{Z}_{(N_x^2)}\\ m\beta(x)\in\mathbb{Z}}} E_{k,\underline{L},mx}(\tau,z). \tag{2.44}$$

Proof. Note that, if $m\beta(x)\in\mathbb{Z}$, then $\beta(mx) = m^2\beta(x)\in\mathbb{Z}$ and hence $mx\in\mathrm{Iso}(D_{\underline{L}})$. Thus, the right-hand side of (2.44) is well-defined. A similar argument to the one used to obtain (2.9) implies that

$$E_{k,\underline{L},r}(\tau,z) = \frac{1}{2}\sum_{A\in\Gamma_\infty\backslash\Gamma} 1|_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\tilde{A}(\tau)\sum_{y\in L^\#/L}\rho_{\underline{L}}(\tilde{A})_{r,y}\vartheta_{\underline{L},y}(\tau,z). \tag{2.45}$$

The fact that $\rho_{\underline{L}}$ is unitary implies that $\rho_{\underline{L}}^*(\tilde{A})^{-1} = \rho_{\underline{L}}(\tilde{A})^t$ and therefore

$$\mathrm{Av}_x\,E_{k,\underline{L},0}(\tau,z) = \frac{1}{N_x^2}\sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2}\varphi^{-1}\sigma_x^*(m,n,0)\frac{1}{2}\sum_{A\in\Gamma_\infty\backslash\Gamma}1|_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\tilde{A}(\tau)\rho_{\underline{L}}^*(\tilde{A})^{-1}\mathfrak{e}_0.$$

Equation (1.27) implies that

$$\begin{aligned}\sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2} \sigma_x^*(m,n,0)\rho_{\underline{L}}^*(\tilde{A})^{-1}\mathfrak{e}_0 &= \sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2} \rho_{\underline{L}}^*(\tilde{A})^{-1}\sigma_x^*\left((m,n,0)^{A^{-1}}\right)\mathfrak{e}_0 \\ &= \sum_{(m',n')\in(\mathbb{Z}_{(N_x^2)})^2} \rho_{\underline{L}}^*(\tilde{A})^{-1}\sigma_x^*(m',n',0)\,\mathfrak{e}_0 \\ &= \sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2} e(mn\beta(x))\rho_{\underline{L}}^*(\tilde{A})^{-1}\mathfrak{e}_{-mx} \\ &= \sum_{m\in\mathbb{Z}_{(N_x^2)}} \rho_{\underline{L}}^*(\tilde{A})^{-1}\mathfrak{e}_{-mx}\sum_{n\in\mathbb{Z}_{(N_x^2)}} e^n(m\beta(x)).\end{aligned}$$

In the second line, we have substituted (m',n') for $\left((m,n)A^{-1}\right)$. The inner sum in the last line is equal to N_x^2 if $m\beta(x)\in\mathbb{Z}$ and to zero otherwise. Thus,

$$\begin{aligned}\mathrm{Av}_x\, E_{k,\underline{L},0}(\tau,z) &= \sum_{\substack{m\in\mathbb{Z}_{(N_x^2)}\\ m\beta(x)\in\mathbb{Z}}} \varphi^{-1}\frac{1}{2}\sum_{A\in\Gamma_\infty\backslash\Gamma} 1|_{k-\frac{\mathrm{rk}(\underline{L})}{2}}\tilde{A}(\tau)\rho_{\underline{L}}(\tilde{A})^t\mathfrak{e}_{-mx} \\ &= \sum_{\substack{m\in\mathbb{Z}_{(N_x^2)}\\ m\beta(x)\in\mathbb{Z}}} E_{k,\underline{L},-mx}(\tau,z) = \sum_{\substack{m\in\mathbb{Z}_{(N_x^2)}\\ m\beta(x)\in\mathbb{Z}}} E_{k,\underline{L},mx}(\tau,z). \qquad \square\end{aligned}$$

Note that both N_x^2 and m are multiples of $\mathrm{lev}(x)$. Set $M_x := \frac{N_x^2}{\mathrm{lev}(x)} - 1$; then the conditions in the above summation can be re-written as

$$\mathrm{Av}_x\, E_{k,\underline{L},0}(\tau,z) = \sum_{j=0}^{M_x} E_{k,\underline{L},j\,\mathrm{lev}(x)x}. \tag{2.46}$$

When k is odd, equation (1.15) asserts that $E_{k,\underline{L},x} = -E_{k,\underline{L},-x}$; on the right hand-side of (2.44), every element m in $\mathbb{Z}_{(N_x^2)}$ satisfies $m\beta(x)\in\mathbb{Z}$ if and only if $-m\beta(x)\in\mathbb{Z}$ and hence the right-hand side of (2.44) vanishes.

We want to determine whether it is possible to obtain all Eisenstein series on the right-hand side of (2.46) without inputting an isotropic element on the left-hand side. In other words, for every r in $\mathrm{Iso}(D_{\underline{L}})$, does there exist an x in $(L^\#/L)\setminus\mathrm{Iso}(D_{\underline{L}})$ such that

$$\mathrm{Av}_x\, E_{k,\underline{L},0} = E_{k,\underline{L},r} + \sum_{s\neq r} E_{k,\underline{L},s}?$$

We give an example where the answer is no. We remind the reader that $D_{\underline{L}}$ is a finite quadratic module. Suppose that $D_{\underline{L}} \simeq A_{p^n}^t$ for some odd prime p, some even positive integer n and some integer t which is coprime to p (see Theorem 1.12). Then $tr^2/p^n\in\mathbb{Z}$ if an only if $p^n \mid r^2$ and therefore

$$\mathrm{Iso}(A_{p^n}^t) = \left\{sp^{\frac{n}{2}} : s\in\{0,1,\dots,p^{\frac{n}{2}}-1\}\right\}.$$

Each non-isotropic element of $A_{p^n}^t$ is equal to $sp^{\frac{n}{2}-l}$, for some positive integer s which is coprime to p and some positive integer l such that $l\le\frac{n}{2}$. Such an element has level equal to p^{2l} and hence every $j\,\mathrm{lev}(x)x$ on the right-hand side of (2.46) is a multiple of $p^{\frac{n}{2}+l}$ and hence, for example, the isotropic element $p^{\frac{n}{2}}$ is never achieved on the right-hand side.

For the remainder of this section, suppose that k is even. Let $h_{r,y}$ denote the y-th component of the theta expansion of $E_{k,\underline{L},r}$, as given in (2.45):

$$h_{r,y}(\tau) = \frac{1}{2} \sum_{A \in \Gamma_\infty \backslash \Gamma} \rho_{\underline{L}}(\tilde{A})_{r,y} 1|_{k-\frac{\mathrm{rk}(L)}{2}} \tilde{A}(\tau).$$

Taking x to be an isotropic element in Proposition 2.24 leads to the main result of this section:

Proposition 2.25. *Suppose that k is even and that $x \in \mathrm{Iso}(D_{\underline{L}})$. Then*

$$\sum_{m \in \mathbb{Z}_{(N_x)}} E_{k,\underline{L},mx}(\tau, z) = \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} \Big(\sum_{m \in \mathbb{Z}_{(N_x)}} h_{0,y+mx}(\tau) \Big) \vartheta_{\underline{L},y}(\tau, z), \tag{2.47}$$

which implies the following identity:

$$\sum_{m \in \mathbb{Z}_{(N_x)}} G_{k,\underline{L},mx}(D, y) = \begin{cases} \sum_{m \in \mathbb{Z}_{(N_x)}} G_{k,\underline{L},0}(D, y + xm), & \text{if } \beta(x, y) \in \mathbb{Z} \text{ and} \\ 0, & \text{otherwise}. \end{cases}$$

Proof. Insert the definition of Av_x on the left-hand side of (2.44) and expand:

$$\begin{aligned} \mathrm{Av}_x\, E_{k,\underline{L},0}(\tau, z) =& \frac{1}{N_x^2} \varphi^{-1} \sum_{(m,n) \in (\mathbb{Z}_{(N_x^2)})^2} \sum_{y \in L^\#/L} e(mn\beta(x) - n\beta(x, y)) h_{0,y}(\tau) \mathfrak{e}_{y-mx} \\ =& \frac{1}{N_x^2} \varphi^{-1} \sum_{y \in L^\#/L} h_{0,y}(\tau) \sum_{m \in \mathbb{Z}_{(N_x^2)}} \mathfrak{e}_{y-mx} \sum_{n \in \mathbb{Z}_{(N_x^2)}} e(-n\beta(x, y)) \\ =& \varphi^{-1} \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} h_{0,y}(\tau) \sum_{m \in \mathbb{Z}_{(N_x^2)}} \mathfrak{e}_{y-mx} = N_x \varphi^{-1} \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} h_{0,y}(\tau) \sum_{m \in \mathbb{Z}_{(N_x)}} \mathfrak{e}_{y-mx}, \end{aligned}$$

since $\mathfrak{e}_{y-mx}$ only depends on m mod N_x. Set $y' := y - mx$ and drop the prime from the notation. Then the last line can be written as

$$\begin{aligned} \mathrm{Av}_x\, E_{k,\underline{L},0}(\tau, z) =& N_x \varphi^{-1} \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} \Big(\sum_{m \in \mathbb{Z}_{(N_x)}} h_{0,y+mx}(\tau) \Big) \mathfrak{e}_y \\ =& N_x \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} \Big(\sum_{m \in \mathbb{Z}_{(N_x)}} h_{0,y+mx}(\tau) \Big) \vartheta_{\underline{L},y}(\tau, z). \end{aligned}$$

On the other hand, if $x \in \mathrm{Iso}(D_{\underline{L}})$, then $m\beta(x) \in \mathbb{Z}$ for every m in $\mathbb{Z}_{(N_x^2)}$ and thus the right-hand side of (2.44) is equal to

$$\sum_{m \in \mathbb{Z}_{(N_x^2)}} E_{k,\underline{L},mx}(\tau, z) = N_x \sum_{m \in \mathbb{Z}_{(N_x)}} E_{k,\underline{L},mx}(\tau, z),$$

since $E_{k,\underline{L},r}$ only depends on r mod L. The identity involving the Fourier coefficients follows immediately. □

We list some examples in which Proposition 2.25 can be used to compute the Fourier coefficients of Eisenstein series indexed by elements x in $\mathrm{Iso}(D_{\underline{L}})$ of small order.

Example 2.26. Suppose that x is an isotropic element of order 2. Then Proposition 2.25 implies that

$$E_{k,\underline{L},0}(\tau, z) + E_{k,\underline{L},x}(\tau, z) = \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} \big(h_{0,y} + h_{0,y+x} \big)(\tau) \vartheta_{\underline{L},y}(\tau, z),$$

in other words

$$G_{k,\underline{L},0}(D,y) + G_{k,\underline{L},x}(D,y) = \begin{cases} G_{k,\underline{L},0}(D,y) + G_{k,\underline{L},0}(D,y+x), & \text{if } \beta(x,y) \in \mathbb{Z}, \\ 0, & \text{otherwise,} \end{cases}$$

for every (D, y) in $\operatorname{supp}(\underline{L})$. Hence, the Fourier coefficients of $E_{k,\underline{L},x}$ are given by the formula

$$G_{k,\underline{L},x}(D,y) = \begin{cases} G_{k,\underline{L},0}(D,y+x), & \text{if } \beta(x,y) \in \mathbb{Z} \text{ and} \\ -G_{k,\underline{L},0}(D,y), & \text{otherwise.} \end{cases}$$

Example 2.27. Suppose that x is an isotropic element of order 3. Then Proposition 2.25 implies that

$$E_{k,\underline{L},0} + E_{k,\underline{L},x} + E_{k,\underline{L},2x} = \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} \left(h_{0,y} + h_{0,y+x} + h_{0,y+2x}\right) \vartheta_{\underline{L},y}.$$

The fact that $E_{k,\underline{L},x} = E_{k,\underline{L},-x}$ when k is even implies that the Fourier coefficients of $E_{k,\underline{L},x}$ are given by the formula

$$G_{k,\underline{L},x}(D,y) = \begin{cases} \frac{1}{2}\left(G_{k,\underline{L},0}(D,y+x) + G_{k,\underline{L},0}(D,y+2x)\right), & \text{if } \beta(x,y) \in \mathbb{Z} \text{ and} \\ -\frac{1}{2} G_{k,\underline{L},0}(D,y), & \text{otherwise.} \end{cases}$$

Example 2.28. If x in $\operatorname{Iso}(D_{\underline{L}})$ has order 4, then

$$E_{k,\underline{L},0} + 2E_{k,\underline{L},x} + E_{k,\underline{L},2x} = \sum_{\substack{y \in L^\#/L \\ \beta(x,y) \in \mathbb{Z}}} \left(h_{0,y} + h_{0,y+x} + h_{0,y+2x} + h_{0,y+3x}\right) \vartheta_{\underline{L},y}.$$

Since $2x$ has order 2, Example 2.26 implies that

$$G_{k,\underline{L},2x}(D,y) = \begin{cases} G_{k,\underline{L},0}(D,y+2x), & \text{if } \beta(2x,y) \in \mathbb{Z} \text{ and} \\ -G_{k,\underline{L},0}(D,y), & \text{otherwise.} \end{cases}$$

Note that $\beta(x, y) \in \mathbb{Z}$ implies that $\beta(2x, y) \in \mathbb{Z}$ and hence $G_{k,\underline{L},x}(D, y)$ is equal to

$$\begin{cases} \frac{1}{2}\left(G_{k,\underline{L},0}(D,y+x) + G_{k,\underline{L},0}(D,y+3x)\right), & \text{if } \beta(x,y) \in \mathbb{Z}, \\ -\frac{1}{2}\left(G_{k,\underline{L},0}(D,y) + G_{k,\underline{L},0}(D,y+2x)\right), & \text{if } \beta(2x,y) \in \mathbb{Z} \text{ and } \beta(x,y) \notin \mathbb{Z} \text{ and} \\ 0, & \text{if } \beta(2x,y) \notin \mathbb{Z}. \end{cases}$$

Example 2.29. When x has order 6, Proposition 2.25 gives a formula for

$$E_{k,\underline{L},0} + E_{k,\underline{L},x} + E_{k,\underline{L},2x} + E_{k,\underline{L},3x} + E_{k,\underline{L},4x} + E_{k,\underline{L},5x}.$$

Since $3x$ has order 2, Example 2.26 implies that

$$G_{k,\underline{L},3x}(D,y) = \begin{cases} G_{k,\underline{L},0}(D,y+3x), & \text{if } \beta(3x,y) \in \mathbb{Z} \text{ and} \\ -G_{k,\underline{L},0}(D,y), & \text{otherwise.} \end{cases}$$

Since $2x$ has order 3, Example 2.27 implies that

$$G_{k,\underline{L},2x}(D,y) = \begin{cases} \frac{1}{2}\left(G_{k,\underline{L},0}(D,y+2x) + G_{k,\underline{L},0}(D,y+4x)\right), & \beta(2x,y) \in \mathbb{Z} \text{ and} \\ -\frac{1}{2} G_{k,\underline{L},0}(D,y), & \text{otherwise} \end{cases}$$

and note that $E_{k,\underline{L},4x} = E_{k,\underline{L},2x}$. If $\beta(x, y) \in \mathbb{Z}$, then $\beta(2x, y) \in \mathbb{Z}$ and $\beta(3x, y) \in \mathbb{Z}$. If $\beta(x, y) \notin \mathbb{Z}$ but $\beta(2x, y) \in \mathbb{Z}$, then $\beta(x, y) = a/2$ for some odd a and therefore $\beta(3x, y) \notin$

$\mathbb{Z}$. Similarly, if $\beta(x,y) \notin \mathbb{Z}$ and $\beta(3x,y) \in \mathbb{Z}$, then $\beta(2x,y) \notin \mathbb{Z}$. Since $E_{k,\underline{L},x} = E_{k,\underline{L},5x}$, it follows that $G_{k,\underline{L},x}(D,y)$ is equal to

$$\begin{cases} \frac{1}{2}\left(G_{k,\underline{L},0}(D,y+x) + G_{k,\underline{L},0}(D,y+5x)\right), & \text{if } \beta(x,y) \in \mathbb{Z}, \\ -\frac{1}{2}\left(G_{k,\underline{L},0}(D,y+2x) + G_{k,\underline{L},0}(D,y+4x)\right), & \text{if } \beta(2x,y) \in \mathbb{Z} \text{ and } \beta(x,y) \notin \mathbb{Z}, \\ -\frac{1}{2}G_{k,\underline{L},0}(D,y+3x), & \text{if } \beta(3x,y) \in \mathbb{Z} \text{ and } \beta(x,y) \notin \mathbb{Z}, \\ \frac{1}{2}G_{k,\underline{L},0}(D,y), & \text{if } \beta(2x,y) \text{ and } \beta(3x,y) \notin \mathbb{Z}. \end{cases}$$

If x is an isotropic element of order 5, then we obtain a formula for the Fourier coefficients of $E_{k,\underline{L},x} + E_{k,\underline{L},2x}$:

$$\left(G_{k,\underline{L},x} + G_{k,\underline{L},2x}\right)(D,y) = \begin{cases} \substack{\frac{1}{2}(G_{k,\underline{L},0}(D,y+x)+G_{k,\underline{L},0}(D,y+2x) \\ +G_{k,\underline{L},0}(D,y+3x)+G_{k,\underline{L},0}(D,y+4x)),} & \beta(x,y) \in \mathbb{Z} \text{ and} \\ -\dfrac{1}{2}G_{k,\underline{L},0}(D,y), & \text{otherwise.} \end{cases}$$

In general, if x in $\mathrm{Iso}(D_{\underline{L}})$ has odd prime order p, then we obtain a formula for

$$E_{k,\underline{L},x} + E_{k,\underline{L},2x} + E_{k,\underline{L},3x} + \cdots + E_{k,\underline{L},\frac{p-1}{2}x}.$$

When x has order 8, Proposition 2.25 gives a formula for

$$E_{k,\underline{L},0} + E_{k,\underline{L},x} + E_{k,\underline{L},2x} + E_{k,\underline{L},3x} + E_{k,\underline{L},4x} + E_{k,\underline{L},5x} + E_{k,\underline{L},6x} + E_{k,\underline{L},7x}$$

and we can compute the Fourier coefficients of $E_{k,\underline{L},4x}$ and $E_{k,\underline{L},2x}$ (which is equal to $E_{k,\underline{L},6x}$) using Examples 2.26 and 2.28, respectively. However, we then obtain a formula for the Fourier coefficients of $E_{k,\underline{L},x} + E_{k,\underline{L},3x}$ only. Note that this method resembles a sieving technique.

Let ξ be a primitive character of conductor $F \mid N_x$. We remind the reader of Definition 1.31 of the twisted Eisenstein series,

$$E_{k,\underline{L},x,\xi} = \sum_{d \in \mathbb{Z}_{(N_x)}^\times} \xi(d) E_{k,\underline{L},dx}.$$

This resembles the left-hand side of (2.47). Define

$$\mathrm{Av}_{x,\xi}\,\phi(\tau,z) := \frac{1}{N_x^2} \sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2} \xi(m)\left(\varphi^{-1}\sigma_x^*(m,n,0)\varphi\right)\phi(\tau,z).$$

This expression is independent of the coset representatives of $\mathbb{Z}_{(N_x^2)}$, however

$$\begin{aligned} \varphi(\mathrm{Av}_{x,\xi}\,\phi)|_{k-\frac{\mathrm{rk}(L)}{2}}\tilde{A}(\tau) =& \frac{1}{N_x^2} \sum_{(m,n)\in(\mathbb{Z}_{(N_x^2)})^2} \xi(m)\rho_{\underline{L}}^*(\tilde{A})\sigma_x^*((m,n,0)^A)F(\tau) \\ \neq& \frac{1}{N_x^2} \sum_{(m',n')\in(\mathbb{Z}_{(N_x^2)})^2} \xi(m')\rho_{\underline{L}}^*(\tilde{A})\sigma_x^*(m',n',0)F(\tau). \end{aligned}$$

In other words, if we were to twist Av_x by ξ, then $\mathrm{Av}_{x,\xi}\,\phi$ fails to be modular, due to the fact that the change of variable $(m',n') = (m,n)A$ made in (2.43) does not preserve ξ. We have also made this change of variable in the proof of Proposition 2.24, hence we cannot simply define $\mathrm{Av}_{x,\xi}$ on Eisenstein series directly.

CHAPTER 3

Hecke operators and the action of the orthogonal group

In the future, we would like to establish a precise correspondence between Jacobi forms of lattice index and elliptic modular forms. One of the key ingredients going into the proof of Theorem 1.37 is the theory of newforms developed in [**EZ85**] for Jacobi forms of scalar index. In this chapter, we study Hecke operators and the operators arising from the action of the orthogonal group of the discriminant module associated with the lattice in the index. These families of operators were both defined for the first time in [**Ajo15**]. In the final section, we study the correspondence between Jacobi forms for the root lattices of type D_n (n odd) and elliptic modular forms for small weights.

3.1. Hecke operators and lifting maps

We review the main results in [**Ajo15**]. The reader can consult the cited text for the proofs. Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and let $k \geq \frac{\mathrm{rk}(\underline{L})}{2}$ be an integer.

3.1.1. Definition and properties of Hecke operators. Set

$$\mathbb{N}_{\underline{L}} := \{n \in \mathbb{N} : (n, \mathrm{lev}(\underline{L})) = 1\}.$$

Hecke operators were defined in [**Ajo15**, §2.5] as *double coset operators*:

Definition 3.1. For every l in $\mathbb{N}_{\underline{L}}$, define the following operator on $J_{k,\underline{L}}$:

$$T_0(l)\phi := l^{k-2-rk(\underline{L})} \sum_{g \in J^{\underline{L}} \backslash J^{\underline{L}} \left(\begin{smallmatrix} l^{-1} & 0 \\ 0 & l \end{smallmatrix}\right) J^{\underline{L}}} \phi|_{k,\underline{L}}\, g.$$

The operators $T_0(\cdot)$ are well-defined, in other words, they do not depend on the choice of coset representatives of $J^{\underline{L}} \backslash J^{\underline{L}} \left(\begin{smallmatrix} l^{-1} & 0 \\ 0 & l \end{smallmatrix}\right) J^{\underline{L}}$. Furthermore, they map $J_{k,\underline{L}}$ to itself and they preserve the subspaces of cusp forms and Eisenstein series. This can be seen from their action on the Fourier coefficients of Jacobi forms ([**Ajo15**, Proposition 2.5.6 and Theorem 2.6.8]). They are *primitive* Hecke operators, in the sense that the Γ-component of every set of coset representatives of $J^{\underline{L}} \backslash J^{\underline{L}} \left(\begin{smallmatrix} l^{-1} & 0 \\ 0 & l \end{smallmatrix}\right) J^{\underline{L}}$ is given by *primitive* matrices, i.e. matrices whose entries are coprime.

Definition 3.2 (Hecke operators). For every l in $\mathbb{N}_{\underline{L}}$, define the following operator on $J_{k,\underline{L}}$:

- if $\mathrm{rk}(\underline{L})$ is odd, then

$$T(l)\phi := \sum_{s^2 | l, s>0} s^{2k-\mathrm{rk}(\underline{L})-3} T_0\left(\frac{l}{s^2}\right)\phi; \tag{3.1}$$

- if $\mathrm{rk}(\underline{L})$ is even, then

$$T(l)\phi := \sum_{\substack{d,s>0 \\ sd^2 | l, s \text{ square-free}}} \chi_{\underline{L}}(s)(sd^2)^{k-\frac{\mathrm{rk}(\underline{L})}{2}-2} T_0\left(\frac{l}{sd^2}\right)\phi. \tag{3.2}$$

Equation (3.1) matches the relation between Hecke operators for elliptic modular forms of weight $2k - \mathrm{rk}(\underline{L}) - 1$ and the corresponding operators defined with primitive matrices. For every l in $\mathbb{N}$, set

$$\mathrm{M}(l) := \{A \in \mathrm{M}_2(\mathbb{Z}) : \det(A) = l\}. \tag{3.3}$$

A complete set of coset representatives for $\Gamma\backslash\mathrm{M}(l)$ is given by the set

$$\Delta_l := \{A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, b, d \in \mathbb{Z}, a, d \geq 0, ad = l \text{ and } 0 \leq b < d\}.$$

For every matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $\mathrm{M}_2(\mathbb{Z})$, set $\gcd(A) := \gcd(a, b, c, d)$. Define

$$\Delta_l^{\mathrm{pr}} := \{A \in \Delta_l : \gcd(A) = 1\}.$$

The $|_{k,\underline{L}}$-action of matrices in $\mathrm{GL}_2^+(\mathbb{R})$ on holomorphic, complex-valued functions defined on $\mathfrak{H} \times (L \otimes \mathbb{C})$ is defined as

$$(\phi, M) \mapsto \phi|_{k,\underline{L}} M := \phi|_{k,\underline{L}} \left(\frac{1}{\sqrt{\det(M)}} M \right). \tag{3.4}$$

The following holds:

Lemma ([**Ajo15**, Lemma 2.6.6]). *Let ϕ be an element of $J_{k,\underline{L}}$. For every l in $\mathbb{N}_{\underline{L}}$, the action of $T_0(l)$ on ϕ can be written as*

$$T_0(l)\phi = l^{k-2-2\mathrm{rk}(\underline{L})} \sum_{h \in L^2/lL^2} \sum_{A \in \frac{1}{l}\Delta_{l^2}^{pr}} \phi|_{k,\underline{L}}(A, h). \tag{3.5}$$

Lemma 3.3. *Let $\phi \in J_{k,\underline{L}}$, $l \in \mathbb{N}$ and $s^2 \mid l$. Then*

$$\phi|_{k,\underline{L}}(A, (\lambda, \mu)) = \phi|_{k,\underline{L}} \left(A, \left(\lambda + \frac{l}{s^2} e_n, \mu \right) \right) = \phi|_{k,\underline{L}} \left(A, \left(\lambda, \mu + \frac{l}{s^2} e_n \right) \right) \tag{3.6}$$

for every A in $\frac{s^2}{l}\Delta^{pr}_{\frac{l^2}{s^4}}$ and every n in $\{1, \ldots, \mathrm{rk}(\underline{L})\}$.

Proof. We have

$$\frac{s^2}{l}\Delta^{\mathrm{pr}}_{\frac{l^2}{s^4}} = \frac{s^2}{l} \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}, a, d \geq 0, ad = \frac{l^2}{s^4}, 0 \leq b < d, (a, b, d) = 1 \right\}$$

and therefore

$$\phi|_{k,\underline{L}}(A, (\lambda, \mu))(\tau, z) = \left(\frac{s^2 d}{l} \right)^{-k} e\left(\beta(\lambda)\tau + \beta(\lambda, z)\right) \phi\left(\frac{a\tau + b}{d}, \frac{l(z + \lambda\tau + \mu)}{s^2 d} \right)$$

for some a, b and d as above. On the other hand,

$$\begin{aligned}
\phi|_{k,\underline{L}} \left(A, \left(\lambda + \frac{l}{s^2} e_n, \mu \right) \right)(\tau, z) &= \left(\frac{s^2 d}{l} \right)^{-k} e\left(\beta\left(\lambda + \frac{l}{s^2} e_n \right)\tau + \beta\left(\lambda + \frac{l}{s^2} e_n, z \right) \right) \\
&\quad \times \phi\left(\frac{a\tau + b}{d}, \frac{l\left(z + (\lambda + \frac{l}{s^2} e_n)\tau + \mu\right)}{s^2 d} \right) \\
&= \left(\frac{s^2 d}{l} \right)^{-k} e\left(\beta\left(\lambda + \frac{l}{s^2} e_n \right)\tau + \beta\left(\lambda + \frac{l}{s^2} e_n, z \right) \right) \phi\left(\frac{a\tau + b}{d}, \frac{l(z + \lambda\tau + \mu)}{s^2 d} + \frac{l^2\tau}{s^4 d} e_n \right) \\
&= \left(\frac{s^2 d}{l} \right)^{-k} e\left(\beta\left(\lambda + \frac{l}{s^2} e_n \right)\tau + \beta\left(\lambda + \frac{l}{s^2} e_n, z \right) \right) \phi\left(\tau', z' + d\tau' e_n - b e_n \right),
\end{aligned}$$

where we have made the substitutions $\tau' = \frac{a\tau+b}{d}$ and $z' = \frac{l(z+\lambda\tau+\mu)}{s^2 d}$ and we have used the fact that $ad = \frac{l^2}{s^4}$. Since $\phi \in J_{k,\underline{L}}$ and $de_n, -be_n \in L$, we have

$$\begin{aligned}\phi\left(\tau', z' + d\tau' e_n - be_n\right) =& \phi\left(\tau', z' + d\tau' e_n - be_n\right)|_{\underline{L}}(-de_n, be_n)\\ =& \phi(\tau', z')e(\tau'\beta(de_n) - \beta(de_n, z' + d\tau' e_n - de_n))\\ =& \phi\left(\frac{a\tau+b}{d}, \frac{l(z+\lambda\tau+\mu)}{s^2 d}\right)\\ & \times e\left(-d(a\tau+b)\beta(e_n) - \beta\left(e_n, \frac{l(z+\lambda\tau+\mu)}{s^2}\right)\right)\end{aligned}$$

and we obtain that

$$\phi|_{k,\underline{L}}\left(A, \left(\lambda + \frac{l}{s^2}e_n, \mu\right)\right)(\tau, z) = \phi|_{k,\underline{L}}(A, (\lambda, \mu))(\tau, z),$$

as claimed. We include the calculations in the exponential term for the sake of completeness:

$$\begin{aligned}& e\left(\beta\left(\lambda + \frac{l}{s^2}e_n\right)\tau + \beta\left(\lambda + \frac{l}{s^2}e_n, z\right) - d(a\tau+b)\beta(e_n) - \beta\left(e_n, \frac{l(z+\lambda\tau+\mu)}{s^2}\right)\right)\\ =& e\Bigg(\left(\beta(\lambda) + \beta\left(\frac{l}{s^2}e_n\right) + \beta\left(\lambda, \frac{l}{s^2}e_n\right)\right)\tau + \beta(\lambda, z) + \beta\left(\frac{l}{s^2}e_n, z\right)\\ & - \beta(e_n)ad\tau - \beta(e_n)db - \beta\left(e_n, \frac{lz}{s^2}\right) - \beta\left(e_n, \frac{l}{s^2}\lambda\right)\tau - \beta\left(e_n, \frac{l}{s^2}\mu\right)\Bigg)\\ =& e(\beta(\lambda)\tau + \beta(z, \lambda)).\end{aligned}$$

We also have

$$\begin{aligned}& \phi|_{k,\underline{L}}\left(A, \left(\lambda, \mu + \frac{l}{s^2}e_n\right)\right)(\tau, z)\\ =& \left(\frac{s^2 d}{l}\right)^{-k} e(\beta(\lambda)\tau + \beta(\lambda, z))\phi\left(\frac{a\tau+b}{d}, \frac{l\left(z+\lambda\tau+(\mu+\frac{l}{s^2}e_n)\right)}{s^2 d}\right)\\ =& \left(\frac{s^2 d}{l}\right)^{-k} e(\beta(\lambda)\tau + \beta(\lambda, z))\phi|_{\underline{L}}\left(0, \frac{l^2}{s^4 d}e_n\right)\left(\frac{a\tau+b}{d}, \frac{l(z+\lambda\tau+\mu)}{s^2 d}\right)\\ =& \left(\frac{s^2 d}{l}\right)^{-k} e(\beta(\lambda)\tau + \beta(\lambda, z))\phi\left(\frac{a\tau+b}{d}, \frac{l(z+\lambda\tau+\mu)}{s^2 d}\right),\end{aligned}$$

since $\phi \in J_{k,\underline{L}}$ and $\frac{l^2}{s^4 d}e_n \in L$. $\square$

We use the last two lemmas to obtain a new formula for Hecke operators:

Proposition 3.4. *Let ϕ be an element of $J_{k,\underline{L}}$. For every l in $\mathbb{N}_{\underline{L}}$, the action of $T(l)$ on ϕ can be written as*

$$T(l)\phi = l^{k-2-2\mathrm{rk}(\underline{L})} \sum_{\substack{s^2|l\\ s>0}} s^{1-\mathrm{rk}(\underline{L})} \sum_{A\in\frac{s^2}{l}\Delta^{pr}_{\frac{l^2}{s^4}}} \sum_{h\in L^2/lL^2} \phi|_{k,\underline{L}}(A, h)$$

if $\mathrm{rk}(\underline{L})$ *is odd and as*

$$T(l)\phi = l^{k-2-2\mathrm{rk}(\underline{L})} \sum_{\substack{d,s>0\\ sd^2|l, s \text{ square-free}}} \chi_{\underline{L}}(s)(sd^2)^{\frac{-\mathrm{rk}(\underline{L})}{2}} \sum_{A\in\frac{sd^2}{l}\Delta^{pr}_{\frac{l^2}{s^2d^4}}} \sum_{h\in L^2/lL^2} \phi|_{k,\underline{L}}(A, h)$$

if $\mathrm{rk}(\underline{L})$ *is even.*

Proof. Suppose that $\mathrm{rk}(\underline{L})$ is odd and insert the expression for $T_0(l)$ given in (3.5) into (3.1):

$$T(l)\phi = l^{k-2-2\mathrm{rk}(\underline{L})} \sum_{\substack{s^2|l \\ s>0}} s^{3\mathrm{rk}(\underline{L})+1} \sum_{A\in\frac{s^2}{l}\Delta^{\mathrm{pr}}_{\frac{l^2}{s^4}}} \sum_{h\in L^2/\frac{l}{s^2}L^2} \phi|_{k,\underline{L}}(A,h).$$

Fix a $\mathbb{Z}$-basis $\{e_1,\dots,e_{\mathrm{rk}(\underline{L})}\}$ of L. Two elements λ and λ' of L lie in the same congruence class modulo lL if and only if each of their $\mathrm{rk}(\underline{L})$ coordinates lie in the same congruence class modulo l. It follows that $|L/lL| = l^{\mathrm{rk}(\underline{L})}$ for every l in $\mathbb{N}$. Combining this with (3.6) implies that

$$\sum_{A\in\frac{s^2}{l}\Delta^{\mathrm{pr}}_{\frac{l^2}{s^4}}} \sum_{h\in L^2/\frac{l}{s^2}L^2} \phi|_{k,\underline{L}}(A,h) = s^{-4\mathrm{rk}(\underline{L})} \sum_{A\in\frac{s^2}{l}\Delta^{\mathrm{pr}}_{\frac{l^2}{s^4}}} \sum_{h\in L^2/lL^2} \phi|_{k,\underline{L}}(A,h)$$

for every s such that $s^2 \mid l$. Thus, the proof is complete for odd rank lattices.

When $\mathrm{rk}(\underline{L})$ is even, equations (3.2) and (3.5) imply that

$$T(l)\phi = l^{k-2-2\mathrm{rk}(\underline{L})} \sum_{\substack{d,s>0 \\ sd^2|l,s \text{ square-free}}} \chi_{\underline{L}}(s)(sd^2)^{\frac{3\mathrm{rk}(\underline{L})}{2}} \sum_{A\in\frac{sd^2}{l}\Delta^{\mathrm{pr}}_{\frac{l^2}{s^2d^4}}} \sum_{h\in L^2/\frac{l}{sd^2}L^2} \phi|_{k,\underline{L}}(A,h).$$

The arguments used in the case of odd rank lattices yield the desired result. □

We remind the reader of Definition 1.10 of $\chi_{\underline{L}}(\cdot,\cdot)$. For every D in $\mathbb{Q}_{\le 0}$ such that $\mathrm{lev}(\underline{L})D \in \mathbb{Z}$ and every a in $\mathbb{N}$, define the function

$$\mu_{\underline{L}}(D,a) := \begin{cases} f\chi_{\underline{L}}\left(\frac{D}{f^2},\frac{a}{f^2}\right), & \text{if } (\mathrm{lev}(\underline{L})D,a) = f^2 \text{ for some } f \text{ in } \mathbb{N} \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

The following two theorems describe the action of Hecke operators on the Fourier coefficients of Jacobi forms:

Theorem 3.5 ([**Ajo15**, Thm 2.6.1]). *Let* $\underline{L} = (L,\beta)$ *be a positive-definite, even lattice of odd rank. Let* ϕ *be an element of* $J_{k,\underline{L}}$ *with a Fourier expansion of the form* (1.13)*, let* $l \in \mathbb{N}_{\underline{L}}$ *and let*

$$T(l)\phi(\tau,z) = \sum_{(D,r)\in supp(\underline{L})} C_{T(l)\phi}(D,r)e\left((\beta(r)-D)\tau+\beta(r,z)\right).$$

Then

$$C_{T(l)\phi}(D,r) = \sum_{\substack{a|l^2 \\ a^2|l^2\,\mathrm{lev}(\underline{L})D}} a^{k-\lceil\frac{\mathrm{rk}(\underline{L})}{2}\rceil-1}\mu_{\underline{L}}(D,a)C_\phi\left(\frac{l^2}{a^2}D, la'r\right), \tag{3.7}$$

where, for every a as above, a' is an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L})$.

Theorem 3.6 ([**Ajo15**, Thm 2.6.3]). *Let* $\underline{L} = (L,\beta)$ *be a positive-definite, even lattice of even rank. Let* ϕ *be an element of* $J_{k,\underline{L}}$ *with a Fourier expansion of the form* (1.13)*, let* $l \in \mathbb{N}_{\underline{L}}$ *and let*

$$T(l)\phi(\tau,z) = \sum_{(D,r)\in supp(\underline{L})} C_{T(l)\phi)}(D,r)e((\beta(r)-D)\tau+\beta(r,z)).$$

Then

$$C_{T(l)\phi}(D,r) = \sum_{a|l^2,\mathrm{lev}(\underline{L})D} a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(a)C_\phi\left(\frac{l^2}{a^2}D, la'r\right),$$

where, for every a as above, a' is an integer such that $aa' \equiv 1 \bmod \operatorname{lev}(\underline{L})$.

REMARK 3.7. It is pointed out in [**Ajo15**], if $\operatorname{lev}(r)D$ is square-free, then (3.7) simplifies to

$$C_{T(l)\phi}(D,r) = \sum_{a|l} a^{k-\lceil \frac{\operatorname{rk}(\underline{L})}{2}\rceil -1} \chi_{\underline{L}}(D,a) C_\phi\left(\frac{l^2}{a^2}D, \frac{l}{a}r\right).$$

We include the short proof, which is not given in [**Ajo15**]. Since $a \mid l^2$ and $(l, \operatorname{lev}(\underline{L})) = 1$, we have

$$a^2 \mid l^2 \operatorname{lev}(\underline{L})D \iff a^2 \mid l^2 \operatorname{lev}(r)D.$$

Hence, if $\operatorname{lev}(r)D$ is square-free, then the conditions on a simplify to $a \mid l$. Since $\mu_{\underline{L}}(D,a) = 0$ unless $(\operatorname{lev}(\underline{L})D, a) = 1$, it follows that $\mu_{\underline{L}}(D,a) = \chi_{\underline{L}}(D,a)$. For a' such that $aa' \equiv 1 \bmod \operatorname{lev}(\underline{L})$, we have $la' \equiv \frac{l}{a} \bmod \operatorname{lev}(\underline{L})$ and hence $la'r \equiv \frac{l}{a}r \bmod L$ (since $\operatorname{lev}(\underline{L})L^\# \subseteq L$). Thus, we have $C_\phi\left(\frac{l^2}{a^2}D, la'r\right) = C_\phi\left(\frac{l^2}{a^2}D, \frac{l}{a}r\right)$ and the argument is complete.

The fact that $T_0(l)$ maps $J_{k,\underline{L}}$ to itself for every l combined with the above two theorems imply that the operators $T(\cdot)$ map $J_{k,\underline{L}}$ to itself and that they preserve the subspaces of cusp forms and Eisenstein series. Furthermore, they are Hermitian under the Petersson scalar product. Hecke operators also satisfy the following multiplicative relation, for every m and n in $\mathbb{N}_{\underline{L}}$:

$$T(m)T(n) = \begin{cases} \displaystyle\sum_{d|m,n} d^{2k-\operatorname{rk}(\underline{L})-2} T\left(\frac{mn}{d^2}\right), & \text{if } \operatorname{rk}(\underline{L}) \text{ is odd and} \\ \displaystyle\sum_{d|m^2,n^2} d^{k-\frac{\operatorname{rk}(\underline{L})}{2}-1} \chi_{\underline{L}}(d) T\left(\frac{mn}{d}\right), & \text{if } \operatorname{rk}(\underline{L}) \text{ is even.} \end{cases}$$

In particular, they commute with each other.

DEFINITION 3.8 (Hecke eigenform). An element ϕ of $J_{k,\underline{L}}$ is called a Hecke eigenform if, for every l in $\mathbb{N}_{\underline{L}}$, there exists a constant $\lambda_\phi(l)$ in $\mathbb{C}$ such that $T(l)\phi = \lambda_\phi(l)\phi$.

The notion of "eigenform" is usually applied to cusp forms. However, the following holds:

THEOREM 3.9 ([**Ajo15**, Thm 3.3.18]). *The series $E_{k,\underline{L},r,\chi}$, where r runs through $\mathscr{R}_{\mathrm{Iso}}$ and χ runs through all primitive Dirichlet characters modulo F, with $F \mid N_r$ and $(-1)^k = \chi(-1)$, form a basis of Hecke eigenforms of $J^{\mathrm{Eis}}_{k,\underline{L}}$. More precisely, define*

$$\lambda(l,k,\underline{L},\chi) := \begin{cases} \sigma^{\chi,\overline{\chi}}_{2k-\operatorname{rk}(\underline{L})-2}(l), & \textit{if } \operatorname{rk}(\underline{L}) \textit{ is odd and} \\ \overline{\chi(l)}\sigma^{\chi,\chi_{\underline{L}}}_{k-\frac{\operatorname{rk}(\underline{L})}{2}-1}(l^2), & \textit{if } \operatorname{rk}(\underline{L}) \textit{ is even.} \end{cases}$$

Then

$$T(l)E_{k,\underline{L},r,\chi} = \lambda(l,k,\underline{L},\chi)E_{k,\underline{L},r,\chi}$$

for every l in $\mathbb{N}_{\underline{L}}$.

Note that $\sigma^{\chi,\overline{\chi}}_{2k-\operatorname{rk}(\underline{L})-2}(l) = \chi(l)\sigma_{2k-\operatorname{rk}(\underline{L})-2}(l)$, since χ is a Dirichlet character of modulus dividing $\operatorname{lev}(\underline{L})$ and $(l, \operatorname{lev}(\underline{L})) = 1$.

It is possible to attach an L-series to every Hecke eigenform. Using the multiplicative properties of Hecke operators, this L-series can be written as an Euler product. This fact was used in [**Ajo15**, §2.7] to indicate a correspondence between Jacobi forms and elliptic modular forms.

DEFINITION 3.10 (L-function of a Hecke eigenform). Let ϕ be a Hecke eigenform in $J_{k,\underline{L}}$, such that $T(l)\phi = \lambda_\phi(l)\phi$ for every l in $\mathbb{N}_{\underline{L}}$. The L-series of ϕ in s is defined as

$$L(s,\phi) := \sum_{l\in\mathbb{N}_{\underline{L}}} \lambda_\phi(l) l^{-s}. \tag{3.8}$$

PROPOSITION ([**Ajo15**, Prop 2.7.15]). *If* $\mathrm{rk}(\underline{L})$ *is odd, then* $L(s,\phi)$ *has the following product expansion:*

$$L(s,\phi) = \prod_{p\in\mathbb{N}_{\underline{L}}} \left(1 - p^{-s}\lambda_\phi(p) + p^{2k-\mathrm{rk}(\underline{L})-2-2s}\right)^{-1}.$$

PROPOSITION 3.11 ([**Ajo15**, Prop 2.7.8]). *For each prime number p in $\mathbb{N}_{\underline{L}}$, set*

$$g_\phi(p) := \lambda_\phi(p) - p^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(p).$$

If $\mathrm{rk}(\underline{L})$ *is even, then* $L(s,\phi)$ *has the following product expansion:*

$$L(s,\phi) = \frac{L_{\mathrm{lev}(\underline{L})}(s-k+\frac{\mathrm{rk}(\underline{L})}{2}+1,\chi_{\underline{L}})}{L_{\mathrm{lev}(\underline{L})}(2s-2k+\mathrm{rk}(\underline{L})+2,\chi_{\underline{L}}^2)} \prod_{p\in\mathbb{N}_{\underline{L}}} \left(1 - g_\phi(p)p^{-s} + p^{2(k-\frac{\mathrm{rk}(\underline{L})}{2}-1-s)}\right)^{-1}.$$

The following remarks was made in [**Ajo15**, §2.7]:

REMARK 3.12. If a Jacobi form ϕ of odd rank lattice index lifts to an elliptic modular form f of weight $2k - \mathrm{rk}(\underline{L}) - 1$ with trivial character and suitable level N, then $L(s,\phi)$ should be equal to the L-series of f (up to a finite number of Euler factors). This is indeed consistent with (1.8).

REMARK 3.13. When $\mathrm{rk}(\underline{L})$ is even, we expect that there exist lifting maps from $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(N,\xi\chi_{\underline{L}})$ to $J_{k,\underline{L}}$ (for every Dirichlet character ξ and suitable level N), such that $T(l^2)$ on the elliptic side corresponds to $\xi(l)T(l)$ on the Jacobi side. If f is a Hecke eigenform in $M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(N,\xi\chi_{\underline{L}})$ with eigenvalues $a(l)$, then

$$\sum_{l\in\mathbb{N}_{\underline{L}}} \overline{\xi(l)}a(l^2)l^{-s} = L(s,\phi) \tag{3.9}$$

if we replace $\lambda_\phi(p)$ with $\overline{\xi(p)}a(p^2)$.

We check that (3.9) holds in the following paragraphs. Set $k_2 := k - \frac{\mathrm{rk}(\underline{L})}{2}$ and insert the Euler products of all of the L-functions in the expression for $L(s,\phi)$ given in Proposition 3.11 in order to obtain that

$$L(s,\phi) = \prod_{p\in\mathbb{N}_{\underline{L}}} \frac{1-\chi_{\underline{L}}(p)^2 p^{-2(s-k_2+1)}}{(1-\chi_{\underline{L}}(p)p^{-(s-k_2+1)})(1-g_\phi(p)p^{-s}+p^{-2(s-k_2+1)})}.$$

On the other hand,

$$\sum_{l\in\mathbb{N}_{\underline{L}}} \overline{\xi(l)}a(l^2)l^{-s} = \prod_{p\in\mathbb{N}_{\underline{L}}} \sum_{m\geq 0} a(p^{2m})\overline{\xi(p^m)}p^{-ms}.$$

Let T denote a formal variable and set

$$g(T) := \sum_{m\geq 0} a(p^{2m})T^{2m} \text{ and}$$

$$h(T) := \sum_{m\geq 0} a(p^{2m+1})T^{2m+1}.$$

Set $\alpha := \xi(p)\chi_{\underline{L}}(p)p^{k_2-1}$ for simplicity. Then the power series $F(T) := g(T) + h(T) = \sum_{m\geq 0} a(p^m)T^m$ satisfies

$$F(T) = \frac{1}{1 - a(p)T + \alpha T^2}, \tag{3.10}$$

in view of (1.8) and the fact that $f \in M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(N, \xi\chi_{\underline{L}})$. The Hecke eigenvalues $a(n)$ satisfy the following recurrence relation for every $m \geq 2$:

$$a(p^m) = a(p)a(p^{m-1}) - \alpha a(p^{m-2}).$$

It follows that

$$\begin{aligned} g(T) - 1 = &\sum_{m\geq 1} a(p^{2m})T^{2m} = a(p)T \sum_{m\geq 1} a(p^{2m-1})T^{2m-1} - \alpha T^2 \sum_{m\geq 1} a(p^{2m-2})T^{2m-2} \\ =&a(p)Th(T) - \alpha T^2 g(T) \\ \iff g(T) =&\frac{1 + a(p)TF(T)}{1 + a(p)T + \alpha T^2} \end{aligned}$$

after rearranging and using the fact that $h(T) = F(T) - g(T)$. Equation (3.10) implies that

$$g(T) = \frac{1 + \alpha T^2}{(1 + \alpha T^2)^2 - a(p)^2 T^2}$$

and, substituting $T^2 = \overline{\zeta(p)}p^{-s}$ in the above, we obtain that

$$\begin{aligned} \sum_{m\geq 0} a(p^{2m})\overline{\xi(p^m)}p^{-ms} =&\frac{1 + \chi_{\underline{L}}(p)p^{k_2-1-s}}{(1 + \chi_{\underline{L}}(p)p^{k_2-1-s})^2 - a(p^2)\overline{\xi(p)}p^{-s}} \\ =&\frac{1 + \chi_{\underline{L}}(p)p^{k_2-1-s}}{1 - \left(a(p)^2\overline{\xi(p)} - 2\chi_{\underline{L}}(p)p^{k_2-1}\right)p^{-s} + p^{-2(s-k_2+1)}} \end{aligned}$$

Since $a(p^2) = a(p)^2 - \alpha$,

$$a(p)^2\overline{\xi(p)} - 2\chi_{\underline{L}}(p)p^{k_2-1} = \lambda_\phi(p) - p^{k_2-1}\chi_{\underline{L}}(p) = g_\phi(p),$$

with $\lambda_\phi(p) = \overline{\xi(p)}a(p^2)$. Hence, equation (3.9) holds.

It is well-known that the Eisenstein subspace $E_k(N, \varepsilon)$ of elliptic modular forms of weight $k > 2$ with character ε for $\Gamma_0(N)$ is spanned by twisted Eisenstein series (see [**Ste07**, Theorem 5.9], for example). Let χ and ψ be primitive Dirichlet characters with conductors L and R, respectively, such that $LR \mid N$ and $\chi(-1)\psi(-1) = (-1)^k$, and define the *twisted Eisenstein series*

$$E_{k,\chi,\psi}(\tau) := c_0 + \sum_{n\geq 1} \sigma_{k-1}^{\chi,\psi}(n)q^n,$$

where

$$c_0 := \begin{cases} 0, & \text{if } L > 1 \text{ and} \\ -\frac{B_{k,\psi}}{2k}, & \text{otherwise.} \end{cases}$$

The generalized Bernoulli numbers $B_{k,\psi}$ are defined by the following identity:

$$\sum_{n=1}^{R} \frac{\psi(n)xe^{nx}}{e^{Rx} - 1} = \sum_{k=0}^{\infty} B_{k,\psi}\frac{x^k}{k!}.$$

The series $E_{k,\chi,\psi}(t\tau)$ (with $LRt \mid N$ and $\chi\psi = \varepsilon$) form a basis of $E_k(N, \varepsilon)$. It follows that Theorem 3.9 is consistent with Remarks 3.12 and 3.13.

3.1.2. Lifting maps. Suppose that $\underline{L}$ is a positive-definite, even lattice of odd rank. The following maps were defined in [**Ajo15**, §4.1] on the space $S_{k,\underline{L}}$:

DEFINITION 3.14. For every ϕ in $S_{k,\underline{L}}$ with Fourier expansion (1.13), x in $L^\#$ and D in $\mathbb{Q}_{\leq 0}$ such that $D \equiv \beta(x) \bmod \mathbb{Z}$, set

$$\mathscr{S}_{D,x}(\phi)(\tau) := \sum_{l=1}^{\infty} \left\{ \sum_{a \mid l} a^{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2} \rceil -1} \chi_{\underline{L}}(D,a) C_\phi\left(\frac{l^2}{a^2}D, \frac{l}{a}x\right)\right\} q^l$$

and, for every D_0 in $\mathbb{Q}_{\leq 0}$, set

$$\mathscr{S}^{\xi_x}_{D_0,x}(\phi)(\tau) := \sum_{\substack{u \bmod N_x \\ D_0 \equiv \beta(ux) \bmod \mathbb{Z}}} \xi_x(u)(\mathscr{S}_{D_0,ux}(\phi) \otimes \xi_x)(w),$$

where $\xi_x(\cdot) = \left(\frac{(-1)^k N_x^2}{\cdot}\right)$ and $\mathscr{S}_{D,x}(\phi) \otimes \xi_x$ denotes the function obtained from $\mathscr{S}_{D,x}(\phi)$ by multiplying its l-th Fourier coefficient with $\xi_x(l)$ for every l.

We recover the scalar index lifting maps $\mathscr{S}_{\Delta,s}$ from Theorem 1.38 by substituting $x = \frac{s}{2m}$ and $D = \frac{\Delta}{4m}$ in Definition 3.14. The following holds:

THEOREM 3.15 ([**Ajo15**, Thm 4.1.4]). *Assume that* $2k - \mathrm{rk}(\underline{L}) \geq 3$ *and that* $N_x^2 D_0$ *is a square-free, negative integer. Then* $\mathscr{S}^{\xi_x}_{D_0,x}$ *maps* $S_{k,\underline{L}}$ *to* $M_{2k-\mathrm{rk}(\underline{L})-1}\left(\frac{\mathrm{lev}(\underline{L})N_x^2}{2}\right)$ *and, if* $2k - \mathrm{rk}(\underline{L}) > 3$, *then it maps cusp forms to cusp forms. Moreover, we have*

$$T(p)\mathscr{S}^{\xi_x}_{D_0,x}(\phi) = \zeta(p)\mathscr{S}^{\xi_x}_{D_0,x}(T(p)\phi),$$

for all primes p *in* $\mathbb{N}_{\underline{L}}$.

The proof of this theorem relies on the connection between Jacobi forms of odd rank lattice index and modular forms of half-integral weight given by theta expansions (1.19). The lifting maps are induced by the Shimura correspondence between half-integral weight and integral weight elliptic modular forms.

REMARK 3.16. We remind the reader that the space $\mathfrak{M}^\varepsilon_k(m)$ was defined in Subsection 1.3.1. It was conjectured in [**Ajo15**, §6.1.1] that, when $\mathrm{rk}(\underline{L})$ is odd, there exists a Hecke-equivariant isomorhism $J_{k,\underline{L}} \xrightarrow{\sim} \mathfrak{M}^\varepsilon_{2k-\mathrm{rk}(\underline{L})-1}(\mathrm{lev}(\underline{L})/4)$, where ε is $-$ if $\mathrm{rk}(\underline{L}) \equiv 1$ or 3 modulo 8 and ε is + otherwise. We verify this conjecture on an example in Section 3.3.

We remind the reader of Definition 1.18 of stably isomorphic lattices.

THEOREM ([**Ajo15**, Thm 4.2.4]). *If the odd rank lattice* $\underline{L}$ *is stably isomorphic to the lattice* $(\mathbb{Z}, (x,y) \mapsto \det(\underline{L})xy)$, *then there exists a Hecke-equivariant isomorphism*

$$J_{k,\underline{L}} \xrightarrow{\sim} \mathfrak{M}^-_{2k-\mathrm{rk}(\underline{L})-1}(\mathrm{lev}(\underline{L})/4).$$

This result is a consequence of Theorems 1.37 and 1.32 (note that the isomorphism in Theorem 1.32 commutes with the action of Hecke operators). In general, the following holds:

LEMMA 3.17. *Let* $\underline{L}_1$ *and* $\underline{L}_2$ *be positive-definite, even lattices of odd rank which are stably isomorphic. Let* j *denote the isomorphism between* $D_{\underline{L}_1}$ *and* $D_{\underline{L}_2}$ *and let* I_j *denote the isomorphism between* $J_{k+\lceil \frac{rk(\underline{L}_2)}{2} \rceil, \underline{L}_2}$ *and* $J_{k+\lceil \frac{rk(\underline{L}_1)}{2} \rceil, \underline{L}_1}$ *from Theorem 1.32. Let* $\phi \in J_{k+\lceil \frac{rk(\underline{L}_2)}{2} \rceil, \underline{L}_2}$ *and let* $x \in L_2^\#$ *and* $D_0 \in \mathbb{Q}_{\leq 0}$ *such that* $N_x^2 D$ *is a square-free integer. Then*

$$\mathscr{S}^{\xi_x}_{D_0,j^{-1}(x)}(I_j(\phi)) = \mathscr{S}^{\xi_x}_{D_0,x}(\phi).$$

Proof. First, note that

$$2\left(k+\lceil\frac{\mathrm{rk}(\underline{L}_1)}{2}\rceil\right)-\mathrm{rk}(\underline{L}_1)-1=2\left(k+\lceil\frac{\mathrm{rk}(\underline{L}_2)}{2}\rceil\right)-\mathrm{rk}(\underline{L}_2)-1=2k$$

and that j preserves the orders of elements and the levels and determinants of the lattices, since it is an isomorphism. We have

$$\begin{aligned}\mathscr{S}_{D,j^{-1}(r)}(I_j(\phi))(\tau)&=\sum_{l=1}^{\infty}\left\{a^{2k}\chi_{\underline{L}_1}(D,a)C_{I_j(\phi)}\left(\frac{l^2}{a^2}D,\frac{l}{a}j^{-1}(r)\right)\right\}q^l\\&=\sum_{l=1}^{\infty}\left\{a^{2k}\chi_{\underline{L}_2}(D,a)C_{\phi}\left(\frac{l^2}{a^2}D,\frac{l}{a}j\circ j^{-1}(r)\right)\right\}q^l=\mathscr{S}_{D,r}(\phi)\end{aligned}$$

for every r in $L_2^\#$ and every D in $\mathbb{Q}_{\leq 0}$ such that $\beta_2(r)\equiv D \bmod \mathbb{Z}$. The result follows. □

For the remainder of this subsection, suppose that $\underline{L}$ is a positive-definite, even lattice of even rank. Furthermore, assume that $\det(\underline{L})=p$ for some odd prime p. In this case, we have $(-1)^{\frac{\mathrm{rk}(\underline{L})}{2}}p\equiv 1 \bmod 4$ and the map

$$a\mapsto\chi(a):=\chi_{\underline{L}}(a)=\left(\tfrac{a}{p}\right)$$

defines a Dirichlet character modulo p. Furthermore, there exists an isomorphism

$$j:D_{\underline{L}}\xrightarrow{\sim}\left(\mathbb{Z}_{(p)},x\mapsto\frac{\alpha x^2}{p}\right)$$

for some integer α which is coprime to p.

Definition 3.18. For every t in $\{\pm 1\}$ and every positive integer k, define the subspace $M_k^t(p,\chi)$ of elliptic modular forms f of weight k for $\Gamma_0(p)$ with nebentypus χ whose Fourier expansions is of the form

$$f(\tau)=\sum_{\substack{n\geq 0\\ \chi(-n)\neq -t}}a_f(n)q^n.$$

Theorem ([**Ajo15**, Thm 5.1.2]). *Let k be an even positive integer, set $k_2:=k-\frac{\mathrm{rk}(\underline{L})}{2}$ and let W_p denote the p-th Fricke involution. Then the maps*

$$\phi\mapsto(-1)^{\frac{\mathrm{rk}(\underline{L})}{2}}h_{\phi,0}|_{k_2}W_p$$

and

$$f\mapsto\frac{1}{2}\sum_{A\in\Gamma_0(p)\backslash SL_2(\mathbb{Z})}\theta_{f,\underline{L}}|_{k,\underline{L}}A,$$

where

$$\theta_{f,\underline{L}}(\tau,z):=(f|_{k_2}W_p)(\tau)\vartheta_{\underline{L},0}(\tau,z),$$

define maps $\mathscr{S}:J_{k,\underline{L}}\to M_{k_2}^{\chi(\alpha)}(p,\chi)$ and $\mathscr{S}^*:M_{k_2}^{\chi(\alpha)}(p,\chi)\to J_{k,\underline{L}}$ which are mutually inverse isomorphisms.

This theorem agrees with Remark 3.13. The Fourier expansion of $\mathscr{S}(\phi)$ is given in [**Ajo15**, §5.1]:

$$\mathscr{S}(\phi)(\tau)=i^{\frac{-\mathrm{rk}(\underline{L})}{2}}p^{\frac{k_2-1}{2}}\sum_{l\geq 0}\left\{\sum_{\substack{x\in L^\#/L\\ \frac{-l}{p}\equiv\beta(x)\bmod\mathbb{Z}}}C_\phi\left(\frac{-l}{p},x\right)\right\}q^l. \tag{3.11}$$

In fact, the following holds:

Proposition 3.19. *The modular form $\mathscr{S}(\phi)$ has the following Fourier expansion:*

$$\mathscr{S}(\phi)(\tau) = 2i^{\frac{-\mathrm{rk}(\underline{L})}{2}} p^{\frac{k_2-1}{2}} \sum_{l\geq 0} C_\phi\left(\frac{-l}{p}, x_l\right) q^l,$$

for some x_l in $L^\#/L$ such that $\beta(x_l) \equiv \frac{-l}{p} \bmod \mathbb{Z}$ and $C_\phi(\cdot,\cdot) = 0$ when no such x_l exists.

Proof. Consider the Fourier expansion (3.11) of $\mathscr{S}(\phi)$. Fix l and suppose that x and y are elements of $L^\#/L$ such that $x \neq y$ and $\beta(x) \equiv \beta(y) \equiv \frac{-l}{p} \bmod \mathbb{Z}$. We remind the reader that there exists an isomorphism $j : D_{\underline{L}} \xrightarrow{\sim} \left(\mathbb{Z}_{(p)}, x \mapsto \frac{\alpha x^2}{p}\right)$. Set $X = j(x)$ and $Y = j(y)$. We have

$$\frac{\alpha X^2}{p} \equiv \frac{\alpha Y^2}{p} \bmod \mathbb{Z} \iff \alpha(X^2 - Y^2) = tp$$

for some integer t. Since $(\alpha, p) = 1$, it follows that $p \mid X^2 - Y^2$, implying that $p \mid (X - Y)$ or $p \mid (X + Y)$. Since $x \neq y$, it follows that $\beta(x) \equiv \beta(y) \bmod \mathbb{Z}$ if and only if $x \equiv (-y) \bmod L$. Thus,

$$\mathscr{S}(\phi)(\tau) = i^{\frac{-\mathrm{rk}(\underline{L})}{2}} p^{\frac{k_2-1}{2}} \sum_{l\geq 0} \left\{ C_\phi\left(\frac{-l}{p}, x_l\right) + C_\phi\left(\frac{-l}{p}, -x_l\right)\right\} q^l,$$

for some $x_l \in L^\#/L$ such that $\beta(x_l) \equiv \frac{-l}{p} \bmod \mathbb{Z}$ and where the Fourier coefficients are equal to zero for l such that no such x_l exists (the latter happens if and only if $\left(\frac{-\alpha l}{p}\right) = -1$). Since $C_\phi(D, r) = (-1)^k C_\phi(D, -r)$ and k is even, we obtain the desired result. $\square$

It was shown in [**Ajo15**, §5.1] that

$$T(n)\mathscr{S}^*(f) = \mathscr{S}^*(T(n^2)f)$$

for every n in $\mathbb{N}_{\underline{L}}$. Since $\mathscr{S}$ and $\mathscr{S}^*$ are mutually inverse isomorphisms, this implies that

$$T(n^2)\mathscr{S}(\phi) = \mathscr{S}(T(n)\phi)$$

for ever n as above.

3.2. The action of the orthogonal group

We remind the reader that, if $\underline{L} = (L, \beta)$ is a positive-definite, even lattice, then its discriminant module $D_{\underline{L}} = (L^\#/L, \beta \bmod \mathbb{Z})$ is a finite quadratic module (Definition 1.11). The following operators acting on $J_{k,\underline{L}}$ were defined in [**Ajo15**, §3.1]:

Proposition 3.20. *The orthogonal group $O(D_{\underline{L}})$ acts on $J_{k,\underline{L}}$ from the right in the following way:*

$$(s, \phi) \mapsto \phi W(s),$$

where, if ϕ in $J_{k,\underline{L}}$ has theta expansion

$$\phi(\tau, z) = \sum_{x\in L^\#/L} h_{\phi,x}(\tau)\vartheta_{\underline{L},x}(\tau, z),$$

then

$$\phi W(s)(\tau, z) := \sum_{x\in L^\#/L} h_{\phi,s(x)}(\tau)\vartheta_{\underline{L},x}(\tau, z).$$

PROOF. Clearly, $\phi W(e) = \phi$ for every ϕ in $J_{k,\underline{L}}$, where e denotes the identity element in $O(D_{\underline{L}})$. Furthermore, for every s_1 and s_2 in $O(D_{\underline{L}})$ we have

$$\begin{aligned}\phi W(s_1 \circ s_2)(\tau, z) &= \sum_{x \in L^\#/L} h_{\phi, s_1 \circ s_2(x)}(\tau) \vartheta_{\underline{L},x}(\tau, z) \\ &= \sum_{x \in L^\#/L} h_{\phi W(s_1), s_2(x)}(\tau) \vartheta_{\underline{L},x}(\tau, z) = \phi W(s_1) \circ W(s_2)(\tau, z). \qquad \square\end{aligned}$$

REMARK 3.21. In [**Ajo15**, Proposition 3.1.1], the author claims that $O(D_{\underline{L}})$ acts from the left on $J_{k,\underline{L}}$. This is not the case, as can be seen above.

It follows that $\phi W(s)$ has the following Fourier expansion:

$$\phi W(s)(\tau, z) = \sum_{(D,r) \in \operatorname{supp}(\underline{L})} C_\phi (D, s(r))\, e\left((\beta(r) - D)\tau + \beta(z, r)\right). \tag{3.12}$$

In particular, the operators $W(\cdot)$ preserve cusp forms and Eisenstein series.

PROPOSITION 3.22. *The operators $W(s)$ are unitary with respect to the Petersson scalar product. In other words, if ϕ and ψ are elements of $J_{k,\underline{L}}$ such that at least one of them is a cusp form, then*

$$\langle \phi W(s), \psi \rangle = \langle \phi, \psi W(s)^{-1} \rangle. \tag{3.13}$$

PROOF. Suppose that $\psi(\tau, z)$ has a theta expansion of the form

$$\psi(\tau, z) = \sum_{x \in L^\#/L} h_{\psi,x}(\tau) \vartheta_{\underline{L},x}(\tau, z)$$

and apply Proposition 1.34 to the left-hand side of (3.13):

$$\begin{aligned}\langle \phi W(s), \psi \rangle &= 2^{-\frac{\operatorname{rk}(\underline{L})}{2}} \det(\underline{L})^{-\frac{1}{2}} \int_{\Gamma \backslash \mathfrak{H}} \sum_{x \in L^\#/L} h_{\phi, s(x)}(\tau) \overline{h_{\psi,x}(\tau)} v^{k - \frac{\operatorname{rk}(\underline{L})}{2} - 2} du dv \\ &= 2^{-\frac{\operatorname{rk}(\underline{L})}{2}} \det(\underline{L})^{-\frac{1}{2}} \int_{\Gamma \backslash \mathfrak{H}} \sum_{x \in L^\#/L} h_{\phi,x}(\tau) \overline{h_{\psi, s^{-1}(x)}(\tau)} v^{k - \frac{\operatorname{rk}(\underline{L})}{2} - 2} du dv \\ &= \langle \phi, \psi W(s^{-1}) \rangle = \langle \phi, \psi W(s)^{-1} \rangle,\end{aligned}$$

since

$$\phi W(s) \circ W(s^{-1})(\tau, z) = \phi W(s \circ s^{-1}) = \phi(\tau, z)$$

for every ϕ in $J_{k,\underline{L}}$. $\square$

REMARK 3.23. In the proof of [**Ajo15**, Theorem 3.2.13], the author claims that Proposition 1.34 implies that the operators $W(s)$ are Hermitian. This is not the case in general, since $W(s) = W(s)^{-1} \iff s = s^{-1}$ and not all elements of $O(D_{\underline{L}})$ need satisfy this property. When $s = s^{-1}$, say that s is an *involution*.

It was proved in [**Ajo15**, §3.1] that the action of $O(D_{\underline{L}})$ commutes with Hecke operators, i.e.

$$T(l)(\phi W(s)) = (T(l)\phi) W(s),$$

for all ϕ in $J_{k,\underline{L}}$, l in $\mathbb{N}_{\underline{L}}$ and s in $O(D_{\underline{L}})$. It follows that $W(s)\phi$ and ϕ have the same Hecke eigenvalues if ϕ is a Hecke eigenform. Furthermore, a well-known result from linear algebra implies the following:

THEOREM ([**Ajo15**, Thm 3.2.13]). *The space $S_{k,\underline{L}}$ has a basis of simultaneous eigenforms for all operators $T(l)$ ($l \in \mathbb{N}_{\underline{L}}$) and $W(s)$ ($s \in O(D_{\underline{L}})$).*

Proof. The proof of [**Ajo15**, Theorem 3.2.13] should be slightly different, in light of Remark 3.23. In the lemma from linear algebra which the author quotes, it is enough to consider a sequence of operators which are diagonalizable and commute with each other in order for the result to hold. By the spectral theorem, the operators $W(s)$ are diagonalizable for all s, since they are unitary and therefore normal. □

We analyse the action of $O(D_{\underline{L}})$ on twisted Eisenstein series:

Proposition 3.24. *For every s in $O(D_{\underline{L}})$, the following holds:*

$$E_{k,\underline{L},r,\chi}W(s) = E_{k,\underline{L},s^{-1}(r),\chi}.$$

Proof. We remind the reader that $J^{\mathrm{Eis}}_{k,\underline{L}}$ is invariant under the action of $W(s)$ for all s in $O(D_{\underline{L}})$. On the other hand, Eisenstein series are uniquely determined by their singular terms. Equation (2.17) implies that

$$\begin{aligned}
C_0(E_{k,\underline{L},x}W(s))(\tau,z) =&\frac{1}{2}\Bigg(\sum_{\substack{r\in L^{\#}\\ s(r)\equiv x \bmod L}} +(-1)^k \sum_{\substack{r\in L^{\#}\\ s(r)\equiv -x \bmod L}}\Bigg)e(\beta(r)\tau+\beta(r,z))\\
=&\frac{1}{2}\Bigg(\sum_{\substack{r\in L^{\#}\\ r\equiv s^{-1}(x) \bmod L}} (-1)^k \sum_{\substack{r\in L^{\#}\\ r\equiv -s^{-1}(x) \bmod L}}\Bigg)e(\beta(r)\tau+\beta(r,z))\\
=&\frac{1}{2}\left(\vartheta_{\underline{L},s^{-1}(x)}+(-1)^k\vartheta_{\underline{L},-s^{-1}(x)}\right)(\tau,z)\\
=&C_0(E_{k,\underline{L},s^{-1}(x)})(\tau,z)
\end{aligned}$$

and therefore $E_{k,\underline{L},x}W(s) = E_{k,\underline{L},s^{-1}(x)}$. Since automorphism preserve the orders of elements, we obtain that

$$E_{k,\underline{L},r,\chi}W(s) = \sum_{d\in\mathbb{Z}^{\times}_{(N_r)}} \chi(d)E_{k,\underline{L},s^{-1}(dr)} = E_{k,\underline{L},s^{-1}(r),\chi},$$

as claimed. □

Corollary 3.25. *For every involution s in $O(D_{\underline{L}})$, the following holds:*

$$E_{k,\underline{L},r,\chi}W(s) = E_{k,\underline{L},s(r),\chi}.$$

For the remainder of this section, we investigate the action of specific elements of $O(D_{\underline{L}})$ on Jacobi forms.

It is well-known that multiplication by an integer which is coprime to the order of a finite abelian group A is a group automorphism of A. Since the determinant and the level of a positive-definite, even lattice share the same set of prime divisors, multiplication by f is an element of $\mathrm{Aut}(D_{\underline{L}})$ for every f in $\mathbb{Z}^{\times}_{(\mathrm{lev}(\underline{L}))}$.

Lemma 3.26. *For every f in $\mathbb{Z}^{\times}_{(\mathrm{lev}(\underline{L}))}$, multiplication by f is an element of $O(D_{\underline{L}})$ if and only if $f^2 \equiv 1 \mod \mathrm{lev}(\underline{L})$.*

Proof. If $\beta(fx) = \beta(x)$ for all x in $D_{\underline{L}}$, then (f^2-1) is a multiple of $\mathrm{lev}(\underline{L})$ by definition. Conversely, if (f^2-1) is a multiple of $\mathrm{lev}(\underline{L})$, then $\beta(fx) = \beta(x)$. □

Lemma 3.27. *If $\mathrm{rk}(\underline{L})$ is odd, then f in $\mathbb{Z}^{\times}_{(\mathrm{lev}(\underline{L}))}$ satisfies $f^2 \equiv 1 \bmod \mathrm{lev}(\underline{L})$ if and only if there exists some $n \parallel (\mathrm{lev}(\underline{L})/4)$ such that f is uniquely determined modulo $\mathrm{lev}(\underline{L})/2$ by the modular equations $f \equiv 1 \bmod 2n$ and $f \equiv -1 \bmod (\mathrm{lev}(\underline{L})/2n)$.*

Proof. We remind the reader that $4 \mid \mathrm{lev}(\underline{L})$ when $\mathrm{rk}(\underline{L})$ is odd. On the right-hand side of the "if and only if" statement, solutions are indexed by $n \parallel (\mathrm{lev}(\underline{L})/4)$ and therefore there are $2^{\omega(\mathrm{lev}(\underline{L})/4)}$ solutions. On the left-hand side, a classical result in modular arithmetic asserts that the polynomial equation $x^2 - 1 \equiv 0 \bmod \mathrm{lev}(\underline{L})$ has the following number of solutions:

$$\begin{cases} 2^{\omega(\mathrm{lev}(\underline{L}))}, & \text{if } 4 \parallel \mathrm{lev}(\underline{L}) \text{ and} \\ 2^{\omega(\mathrm{lev}(\underline{L}))+1}, & \text{if } 8 \mid \mathrm{lev}(\underline{L}). \end{cases}$$

If $4 \parallel \mathrm{lev}(\underline{L})$, then $\omega(\mathrm{lev}(\underline{L})/4) = \omega(\mathrm{lev}(\underline{L})) - 1$. If $8 \mid \mathrm{lev}(\underline{L})$, then $\omega(\mathrm{lev}(\underline{L})/4) = \omega(\mathrm{lev}(\underline{L}))$. Hence, there are twice as many solutions on the left-hand side.

Suppose that $f^2 \equiv 1 \mod \mathrm{lev}(\underline{L})$. Then it is straight-forward to check that $f \bmod \mathrm{lev}(\underline{L})$ is odd, say

$$f \equiv 2d + 1 \bmod \mathrm{lev}(\underline{L}). \tag{3.14}$$

If $d = 0$, then $f \equiv 1 \bmod \mathrm{lev}(\underline{L})/2$ and $f \equiv -1 \bmod 2$. When $d > 0$,

$$f^2 \equiv 1 \bmod \mathrm{lev}(\underline{L}) \implies 4d^2 + 4d = 4K\frac{\mathrm{lev}(\underline{L})}{4}$$

for some K in $\mathbb{Z}$. It follows that $d(d+1) = K\,\mathrm{lev}(\underline{L})/4$ and $(d, d+1) = 1$. This induces a decomposition of $\mathrm{lev}(\underline{L})/4$ into $\mathrm{lev}(\underline{L})/4 = \frac{d}{K_1} \cdot \frac{d+1}{K_2}$, with $K_1K_2 = K$ and $\frac{d}{K_1}, \frac{d+1}{K_2} \in \mathbb{Z}$. Clearly $\left(\frac{d}{k_1}, \frac{d+1}{k_2}\right) = 1$ and we can choose $n = \frac{d}{k_1}$. Then $d \equiv 0 \bmod n$ and $d + 1 \equiv 0 \bmod \mathrm{lev}(\underline{L})/4n$ and (3.14) implies that $f \equiv 1 \bmod 2n$ and $f \equiv -1 \bmod \mathrm{lev}(\underline{L})/2n$, as required.

Conversely, let $n \parallel (\mathrm{lev}(\underline{L})/4)$ and set $t := \mathrm{lev}(\underline{L})/(4n)$. By the Chinese remainder theorem, there exists a unique d_n modulo $\mathrm{lev}(\underline{L})/4$ such that

$$\begin{cases} d_n \equiv 0 \bmod n \text{ and} \\ d_n \equiv -1 \bmod t. \end{cases}$$

Set $f_n = 2d_n + 1$. Then $(4, f_n) = 1$ and above modular congruences imply that $(n, f_n) = (t, f_n) = 1$. Hence, $(f_n, \mathrm{lev}(\underline{L})) = 1$ and

$$f_n^2 = 4d_n(d_n + 1) + 1 \equiv 1 \bmod 4nt,$$

i.e. f_n is a solution of the modular equation $f^2 \equiv 1 \bmod \mathrm{lev}(\underline{L})$. Furthermore, f_n is uniquely determined modulo $\mathrm{lev}(\underline{L})/2$ and $f_n \equiv 1 \bmod 2n$ and $f_n \equiv -1 \bmod 2t$.

Note that

$$\left(f + \frac{\mathrm{lev}(\underline{L})}{2}\right)^2 = f^2 + f\,\mathrm{lev}(\underline{L}) + \frac{\mathrm{lev}(\underline{L})^2}{4} \equiv f^2 \bmod \mathrm{lev}(\underline{L})$$

and therefore each $n \parallel \mathrm{lev}(\underline{L})/4$ gives rise to two solutions modulo $\mathrm{lev}(\underline{L})$. □

For every f in $\mathbb{Z}^\times_{(\mathrm{lev}(\underline{L}))}$ such that $f^2 \equiv 1 \bmod \mathrm{lev}(\underline{L})$ and every x in $L^\#/L$, set $s^f(x) := fx$. Then $(s^f)^{-1}(x) = fx = s^f(x)$, in other words s^f is an *involution* in $O(D_{\underline{L}})$. Proposition 3.24 implies that

$$E_{k,\underline{L},r,\chi}W(s^f) = \sum_{d \in \mathbb{Z}^\times_{(N_r)}} \chi(d) E_{k,\underline{L},dfr} = \sum_{e \in \mathbb{Z}^\times_{(N_r)}} \chi(ef) E_{k,\underline{L},er} = \chi(f) E_{k,\underline{L},r,\chi},$$

where we have made the change of variable $df = e$ and we have used the fact that $N_r \mid \mathrm{lev}(\underline{L})$. Note that $\chi(f) \in \{\pm 1\}$.

EXAMPLE 3.28. Let m be a positive integer and consider the scalar lattice $\underline{L}_m = (\mathbb{Z}, (x, y) \mapsto 2mxy)$. For every $t \parallel \mathrm{lev}(\underline{L}_m)/4$ and every ϕ in $J_{k,m}$ with Fourier expansion (1.23), set

$$W_t\phi(\tau, z) := \frac{1}{t} \sum_{r \in \mathbb{Z}^2/t\mathbb{Z}^2} \phi|_{\underline{L}_m} \left(\frac{1}{t} r\right)(\tau, z).$$

Then it was proved in [**Sko88**] that $W_t\phi \in J_{k,m}$ and, furthermore, it has the following Fourier expansion:

$$W_t\phi(\tau, z) = \sum_{\substack{n,r' \in \mathbb{Z} \\ 4mn - r'^2 \geq 0}} b_\phi(n, \lambda_t r')e(n\tau + r'z),$$

where λ_t is the modulo $2m$ uniquely determined integer which satisfies $\lambda_t \equiv 1 \bmod 2t$ and $\lambda_t \equiv -1 \bmod 2m/t$. The operators W_t are called *Atkin–Lehner involutions* in [**SZ88**], because they play the role of Atkin–Lehner involutions for elliptic modular forms on the side of Jacobi forms. More precisely, the following holds:

$$\mathrm{tr}(T(l) \circ W_t, J_{k,m}) = \mathrm{tr}(T(l) \circ W_t, \mathfrak{M}^-_{2k-2}(m)),$$

It was shown in [**Boy15**, §1.2] that the orthogonal groups of cyclic finite quadratic modules over number fields consist entirely of such operators W_t.

Lemma 3.27 implies that $W_t = W(s^{\lambda_t})$ for every $t \parallel m$ and, conversely, that every operator $W(s^f)$ ($f^2 \equiv 1 \bmod 4m$) is equal to W_n for some $n \parallel m$ (the reader can consult the forward direction in the proof of the lemma for the precise recipe for finding n).

3.3. Jacobi forms of index D_n and elliptic modular forms

In this section, we compute Hecke eigenvalues of Jacobi forms of weight k and index D_n for small values of k and odd n and we compare them with those of elliptic modular forms. We remind the reader of the definition of D_n:

$$D_n = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n : x_1 + \cdots + x_n \in 2\mathbb{Z}\}.$$

It is straight-forward to check that

$$D_n^\# = \left\{x : x \in \mathbb{Z}^n \text{ or } x \in \left(\tfrac{1}{2} + \mathbb{Z}\right)^n\right\}$$

and therefore

$$D_n^\#/D_n = \left\{0, e_n, \frac{e_1 + \cdots + e_n}{2}, \frac{e_1 + \cdots + e_{n-1} - e_n}{2}\right\},$$

where $\{e_i\}_i$ denotes the standard basis of $\mathbb{Z}^n$. Thus,

$$D_n^\#/D_n \simeq \begin{cases} \mathbb{Z}/4\mathbb{Z}, & \text{if } n \text{ is odd and} \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}, & \text{if } n \text{ is even.} \end{cases}$$

Suppose that n is odd. Then the discriminant module associated with D_n is isomorphic to

$$\left(\mathbb{Z}/4\mathbb{Z}, r \mapsto \frac{nr^2}{8} \bmod \mathbb{Z}\right)$$

and $\mathrm{lev}(D_n) = 8$. It follows that D_n is stably isomorphic to D_m for every odd m and n such that $n \equiv m \bmod 8$ and, in view of Theorem 1.32, that $J_{k+\lceil\frac{n}{2}\rceil, D_n} \simeq J_{k+\lceil\frac{m}{2}\rceil, D_m}$ for such m and n. Hence, it suffices to consider $n = 1, 3, 5$ and 7 in this subsection.

In the following paragraphs, we introduce some building blocks for Jacobi forms. The Dedekind η-function was defined in (1.6). It is well-known that

$$\eta^3(\tau) = \frac{1}{2}\sum_{n\in\mathbb{Z}}\left(\tfrac{-4}{n}\right) n q^{\frac{n^2}{8}} = \frac{1}{2}\sum_{n\in\mathbb{Z}}(-1)^n(2n+1)q^{\frac{(2n+1)^2}{8}}. \tag{3.15}$$

The scalar Jacobi theta series $\vartheta(\tau, z)$ was defined in (1.25) and the Jacobi theta series $\vartheta_{\mathbb{Z}^n}(\tau, z)$ was defined in (1.28). We remind the reader of the definition of the unimodular lattice E_8 from Example 1.6, (5) and of that of the Jacobi theta series $\vartheta_{E_8}(\tau, z)$ from (1.29). Let E_k ($k \geq 4$) denote the Eisenstein series of weight k for Γ,

$$E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n\geq 1}\sigma_{k-1}(n)q^n,$$

and let E_2 denote the *quasi-modular* Eisenstein series of weight 2 for Γ,

$$E_2(\tau) := 1 - 24\sum_{n\geq 1}\sigma_1(n)q^n.$$

The discriminant modular form, denoted by Δ, is a cusp form of weight 12 for Γ. It has the following Fourier expansion:

$$\Delta(\tau) = \sum_{n\geq 1}\tau(n)q^n,$$

where $\tau(n)$ is the Ramanujan tau function.

The differential operator $\partial : J_{k,\underline{L}} \to J_{k+2,\underline{L}}$ is defined in [**BS19**] for every ϕ with theta expansion (1.19) as

$$\partial\phi(\tau, z) := \sum_{x\in L^\#/L}\left(q\frac{d}{dq}h_{\phi,x}(\tau)\right)\vartheta_{\underline{L},x}(\tau, z) - \frac{1}{12}\left(k - \frac{\mathrm{rk}(\underline{L})}{2}\right)E_2(\tau)\phi(\tau, z). \tag{3.16}$$

If $f(\tau) \in M_{k_1}(\Gamma)$ and $\phi(\tau, z) \in J_{k_2,\underline{L}}$, with Fourier expansions $\sum_{n\geq 0} a_f(n)q^n$ and (1.13), respectively, then it is easy to check that $f(\tau)\phi(\tau, z) \in J_{k_1+k_2,\underline{L}}$ and that

$$f(\tau)\phi(\tau, z) = \sum_{(D,r)\in\mathrm{supp}(\underline{L})}\left(\sum_{n=0}^{\lfloor -D\rfloor} C_\phi(D+n, r)a_f(n)\right)e((\beta(r) - D)\tau + \beta(r, z)). \tag{3.17}$$

For $n = 1, 3, 5$ and 7, let α_n denote the following isometric embedding of D_n into E_8:

$$(x_1, \ldots, x_n) \mapsto (0, \ldots, 0, x_1, \ldots, x_n).$$

This map can be extended in a natural way to the underlying complex spaces. Its pull-back on spaces of Jacobi forms of weight k is the map $\alpha_n^* : J_{k,E_8} \to J_{k,D_n}$,

$$\alpha_n^*\phi(\tau, z) = \phi(\tau, \alpha_n(z)).$$

3.3.1. Generators and their Fourier expansions. The generators of the spaces J_{k,D_n} ($n = 1, 3, 5$ and 7) over the graded ring of modular forms $M_*(1) := \oplus_{k\in\mathbb{Z}}M_k(1)$ were listed in [**BS19**]. We compute their Fourier expansions in this subsection.

We remind the reader that the Jacobi forms ψ_{12-n,D_n} were defined in Example 1.41. Set

$$\begin{aligned} E_{4,D_n} &:= \alpha_n^*\vartheta_{E_8}, \\ E_{6,D_n} &:= \partial E_{4,D_n} \text{ and} \\ E_{8,D_n} &:= \partial E_{6,D_n}. \end{aligned}$$

Let σ_3 denote the following isometric embedding of D_3 into $\mathbb{Z}^4$:

$$(x, y, z) \mapsto \tfrac{1}{2}(x + y - z, x - y + z, -x + y + z, -x - y - z)$$

and denote its pullback on spaces of Jacobi forms of weight k by σ_3^*.

THEOREM 3.29 ([**BS19**]). *The following holds for $n = 1, 3, 5$ and 7:*

$$J_{2k+1,D_n} = M_{2k+n-11}(1)\psi_{12-n,D_n}. \tag{3.18}$$

For $n = 1, 5$ and 7, we have

$$J_{2k,D_n} = M_{2k-4}(1)E_{4,D_n} \oplus M_{2k-6}(1)E_{6,D_n} \oplus M_{2k-8}(1)E_{8,D_n} \tag{3.19}$$

and, lastly,

$$J_{2k,D_3} = M_{2k-4}(1)E_{4,D_3} \oplus M_{2k-6}(1)E_{6,D_3} \oplus M_{2k-8}(1)\eta^{12}\sigma_3^*\vartheta_{\mathbb{Z}^4}. \tag{3.20}$$

By definition,

$$\vartheta_{\mathbb{Z}^n}(\tau, z) = \sum_{r\in\mathbb{Z}^n} \left(\frac{-4}{r_1\dots r_n}\right) e\left(\frac{r_1^2+\dots+r_n^2}{8}\tau + \frac{r_1z_1+\dots+r_nz_n}{2}\right)$$

and therefore, using (3.15),

$$\begin{aligned}
\psi_{12-n,D_n}(\tau, z) =& \frac{1}{2^{8-n}} \sum_{\substack{n_1,\dots,n_{8-n},\\ m_1,\dots,m_n\in\mathbb{Z}}} \left(\frac{-4}{n_1\dots n_{8-n}m_1\dots m_n}\right) n_1 \dots n_{8-n} e\left(\frac{n_1^2+\dots+n_{8-n}^2+m_1^2+\dots+m_n^2}{8}\tau + \frac{m_1z_1+\dots+m_nz_n}{2}\right) \\
=& \sum_{r\in\left(\frac{1}{2}+\mathbb{Z}\right)^n} (-1)^{r_1+\dots+r_n-\frac{n}{2}} \sum_{x\in\left(\frac{1}{2}+\mathbb{Z}\right)^{8-n}} (-1)^{x_1+\dots+x_{8-n}-\frac{8-n}{2}} x_1 \dots x_{8-n} \\
& \times e\left(\left(\frac{(r,r)}{2} + \frac{(x,x)}{2}\right)\tau + (r, z)\right) \\
=& \sum_{\substack{r\in D_n^\#, D\in\mathbb{Q}_{\le 0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} C_{\psi_{12-n,D_n}}(D, r) e\left(\left(\frac{(r,r)}{2} - D\right)\tau + (r, z)\right),
\end{aligned}$$

where

$$C_{\psi_{12-n,D_n}}(D, r) = \begin{cases} 0, & \text{if } r \in \mathbb{Z}^n \text{ and} \\ \displaystyle\sum_{\substack{x\in\left(\frac{1}{2}+\mathbb{Z}\right)^{8-n}\\ -D=\frac{(x,x)}{2}}} (-1)^{r_1+\dots+r_n+x_1+\dots+x_{8-n}} x_1 \dots x_{8-n}, & \text{if } r \in \left(\frac{1}{2} + \mathbb{Z}\right)^n. \end{cases} \tag{3.21}$$

We have made the substitutions $x_i = \frac{n_i}{2}$ and $r_i = \frac{m_i}{2}$. The value of the expression $(-1)^{x_1+\dots+x_n} x_1 \dots x_n$ does not change under the substitution $x_i = -x_i$ and therefore

$$C_{\psi_{12-n,D_n}}(D, r) = \begin{cases} 0, & \text{if } r \in \mathbb{Z}^n \text{ and} \\ 2^{8-n} \displaystyle\sum_{\substack{x\in\left(\frac{1}{2}+\mathbb{N}\right)^{8-n}\\ -D=\frac{(x,x)}{2}}} (-1)^{r_1+\dots+r_n+x_1+\dots+x_{8-n}} x_1 \dots x_{8-n}, & \text{if } r \in \left(\frac{1}{2} + \mathbb{Z}\right)^n. \end{cases}$$

We have

$$\begin{aligned}
E_{4,D_n}(\tau, z) =& \sum_{r\in E_8} e\left(\frac{(r,r)}{2}\tau + r_{8-n+1}z_1 + \dots + r_8z_n\right) \\
=& \sum_{(r_{8-n+1},\dots,r_8)\in\mathbb{Z}^n\cup\left(\frac{1}{2}+\mathbb{Z}\right)^n} \sum_{\substack{(r_1,\dots,r_{8-n})\in\mathbb{R}^8\\ (r_1,\dots,r_8)\in E_8}} e\left(\left(\frac{r_{8-n+1}^2+\dots+r_8^2}{2} + \frac{r_1^2+\dots+r_{8-n}^2}{2}\right)\tau\right) \\
& \times e\left(r_{8-n+1}z_1 + \dots + r_8z_n\right) \\
=& \vartheta_{D_n,0}(\tau, z) + \sum_{\substack{r\in D_n^\#, D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} C_{4,n}(D, r) e\left(\left(\frac{(r,r)}{2} - D\right)\tau + (r, z)\right),
\end{aligned}$$

where

$$C_{4,n}(D,r) := \begin{cases} \#\left\{x \in \mathbb{Z}^{8-n} : -2D = x_1^2 + \cdots + x_{8-n}^2\right\}, & r \in \mathbb{Z}^n \text{ and} \\ \#\left\{x \in \mathbb{Z}^{8-n} : -2D = x_1^2 + x_1 + \cdots + x_{8-n}^2 + x_{8-n} + \frac{8-n}{4}\right\}, & r \in \left(\frac{1}{2} + \mathbb{Z}\right)^n. \end{cases}$$

Equations (3.16) and (3.17) imply that

$$\begin{aligned} E_{6,D_n}(\tau,z) = & \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{\le 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} (-D) C_{4,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right) \\ & - \frac{8-n}{24} \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{\le 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} C_{4,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right) \\ & + (8-n) \sum_{l \ge 1} \sigma_1(l) q^l \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{\le 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} C_{4,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right) \\ = & \frac{n-8}{24} \vartheta_{D_n,0}(\tau,z) + \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{< 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} C_{6,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right), \end{aligned}$$

where

$$C_{6,n}(D,r) := -\left(D + \frac{8-n}{24}\right) C_{4,n}(D,r) + (8-n) \sum_{l=1}^{\lfloor -D \rfloor} C_{4,n}(D+l,r)\sigma_1(l).$$

Equations (3.16) and (3.17) imply that

$$\begin{aligned} E_{8,D_n}(\tau,z) = & \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{\le 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} (-D) C_{6,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right) \\ & - \frac{12-n}{24} \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{\le 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} C_{6,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right) \\ & + (12-n) \sum_{l \ge 1} \sigma_1(l) q^l \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{\le 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} C_{6,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right) \\ = & \frac{(8-n)(12-n)}{576} \vartheta_{D_n,0}(\tau,z) + \sum_{\substack{r \in D_n^\#, D \in \mathbb{Q}_{< 0} \\ \frac{(r,r)}{2} - D \in \mathbb{Z}}} C_{8,n}(D,r) e\left(\left(\tfrac{(r,r)}{2} - D\right)\tau + (r,z)\right), \end{aligned}$$

where

$$C_{8,n}(D,r) := -\left(D + \frac{12-n}{24}\right) C_{6,n}(D,r) + (12-n) \sum_{l=1}^{\lfloor -D \rfloor} C_{6,n}(D+l,r)\sigma_1(l).$$

We have

$$\begin{aligned}
\eta^{12}(\tau)\sigma_3^*\vartheta_{\mathbb{Z}^4}(\tau,z) =& \frac{1}{16}\sum_{\substack{n_1,\dots,n_4,\\ m_1,\dots,m_4\in\mathbb{Z}}} \left(\tfrac{-4}{n_1\dots n_4 m_1\dots m_4}\right) n_1\dots n_4 e\left(\tfrac{n_1^2+\dots+n_4^2+m_1^2+\dots+m_4^2}{8}\tau\right)\\
&\times e\left(\tfrac{m_1+m_2-m_3-m_4}{4}z_1+\tfrac{m_1-m_2+m_3-m_4}{4}z_2+\tfrac{-m_1+m_2+m_3-m_4}{4}z_3\right)\\
=& \sum_{\substack{x_1,\dots,x_4,\\ r_1,\dots,r_4\in\frac{1}{2}+\mathbb{Z}}} (-1)^{x_1+\dots+x_4+r_1+\dots+r_4} x_1\dots x_4 e\left(\tfrac{x_1^2+\dots+x_4^2+r_1^2+\dots+r_4^2}{2}\tau\right)\\
&\times e\left(\tfrac{r_1+r_2-r_3-r_4}{2}z_1+\tfrac{r_1-r_2+r_3-r_4}{2}z_2+\tfrac{-r_1+r_2+r_3-r_4}{2}z_3\right)\\
=& \sum_{\substack{x_1,\dots,x_4,r_4\in\frac{1}{2}+\mathbb{Z}\\ (m_1,m_2,m_3)\in\mathbb{Z}^3 \text{ or } \left(\frac{1}{2}+\mathbb{Z}\right)^3}} (-1)^{x_1+\dots+x_4+2(m_1+m_2+m_3)} x_1\dots x_4\\
&\times e\left(\tfrac{(m_1+m_2+r_4)^2+(m_1+m_3+r_4)^2+(m_2+m_3+r_4)^2}{2}\tau\right)\\
&\times e\left(\tfrac{x_1^2+\dots+x_4^2+r_4^2}{2}\tau+m_1z_1+m_2z_2+m_3z_3\right)\\
=& \sum_{\substack{m\in D_3^\#,D\in\mathbb{Q}_{\le 0}\\ \frac{(m,m)}{2}-D\in\mathbb{Z}}} C_{\psi_{8,D_3}}(D,m)e\left(\left(\tfrac{(m,m)}{2}-D\right)\tau+(m,z)\right),
\end{aligned}$$

where

$$C_{\psi_{8,D_3}}(D,m) = 16\sum_{\substack{x_1,\dots,x_4\in\frac{1}{2}+\mathbb{N},r_4\in\frac{1}{2}+\mathbb{Z}\\ -2D=(m_1+m_2+m_3+2r_4)^2+x_1^2+\dots+x_4^2}} (-1)^{x_1+\dots+x_4+2(m_1+m_2+m_3)} x_1\dots x_4.$$

We have made the substitutions $x_i=\frac{n_i}{2}$, $r_i=\frac{m_i}{2}$ and $m_i=\frac{r_1-r_{4-i}+r_3-r_4}{2}$.

3.3.2. Computation of Fourier coefficients. Equation (3.18) implies that

$$\begin{aligned}
J_{16-n,D_n} &= \mathbb{C}E_4\psi_{12-n,D_n}, & J_{18-n,D_n} &= \mathbb{C}E_6\psi_{12-n,D_n},\\
J_{20-n,D_n} &= \mathbb{C}E_8\psi_{12-n,D_n}, & J_{22-n,D_n} &= \mathbb{C}E_{10}\psi_{12-n,D_n}\\
J_{24-n,D_n} &= \mathbb{C}E_{12}\psi_{12-n,D_n}\oplus\mathbb{C}\Delta\psi_{12-n,D_n}
\end{aligned}$$

and (3.17) implies that

$$\begin{aligned}
E_t(\tau)\psi_{12-n,D_n}(\tau,z) =& \sum_{\substack{r\in D_n^\#,D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \left(C_{\psi_{12-n,D_n}}(D,r)-\frac{2t}{B_t}\sum_{l=1}^{\lfloor -D\rfloor} C_{\psi_{12-n,D_n}}(D+l,r)\sigma_{t-1}(l)\right)\\
&\times e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right),\\
\Delta(\tau)\psi_{12-n,D_n}(\tau,z) =& \sum_{\substack{r\in D_n^\#,D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \sum_{l=1}^{\lfloor -D\rfloor} C_{\psi_{12-n,D_n}}(D+l,r)\tau(l)e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right) \text{ and}
\end{aligned}$$

$$\begin{aligned}
E_t(\tau)E_{4,D_n}(\tau,z) =& \vartheta_{D_n,0}(\tau,z)+\sum_{\substack{r\in D_n^\#,D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \left(C_{4,n}(D,r)-\frac{2t}{B_t}\sum_{l=1}^{\lfloor -D\rfloor} C_{4,n}(D+l,r)\sigma_{t-1}(l)\right)\\
&\times e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right)
\end{aligned}$$

for every positive integer $t \geq 2$. It follows from (3.19) that

$$\begin{aligned}
&E_4(\tau)E_{4,D_n}(\tau,z) - \tfrac{576}{(8-n)(12-n)}E_{8,D_n}(\tau,z)\\
(3.22)\qquad &= \sum_{\substack{r\in D_n^\#, D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \Big(C_{4,n}(D,r) - \tfrac{576}{(8-n)(12-n)}C_{8,n}(D,r) + 240\sum_{l=1}^{\lfloor -D\rfloor} C_{4,n}(D+l,r)\sigma_3(l)\Big)\\
&\times e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau + (r,z)\right)
\end{aligned}$$

is a cusp form in J_{8,D_n} when $n = 1, 5$ or 7. Equation (3.17) implies that

$$\begin{aligned}
E_t(\tau)E_{6,D_n}(\tau,z) = \frac{n-8}{24}\vartheta_{D_n,0}(\tau,z) + &\sum_{\substack{r\in D_n^\#, D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \Big(C_{6,n}(D,r) - \frac{2t}{B_t}\sum_{l=1}^{\lfloor -D\rfloor} C_{6,n}(D+l,r)\sigma_{t-1}(l)\Big)\\
&\times e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau + (r,z)\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{8-n}{24}E_6(\tau)E_{4,D_n}(\tau,z) + E_4(\tau)E_{6,D_n}(\tau,z)\\
(3.23)\qquad &= \sum_{\substack{r\in D_n^\#, D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \Big[\frac{8-n}{24}C_{4,n}(D,r) + C_{6,n}(D,r) - \sum_{l=1}^{\lfloor -D\rfloor}\Big((8-n)21C_{4,n}(D+l,r)\sigma_5(l)\\
&- 240C_{6,n}(D+l,r)\sigma_3(l)\Big)\Big] e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau + (r,z)\right)
\end{aligned}$$

is a cusp form in J_{10,D_n}. Equation (3.17) implies that

$$\begin{aligned}
E_t(\tau)E_{8,D_n}(\tau,z) = &\frac{(8-n)(12-n)}{576}\vartheta_{D_n,0}(\tau,z) + \sum_{\substack{r\in D_n^\#, D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \Big(C_{8,n}(D,r) - \frac{2t}{B_t}\\
&\times \sum_{l=1}^{\lfloor -D\rfloor} C_{8,n}(D+l,r)\sigma_{t-1}(l)\Big) e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau + (r,z)\right).
\end{aligned}$$

It follows from (3.19) that

$$\begin{aligned}
&\frac{8-n}{24}E_8(\tau)E_{4,D_n}(\tau,z) + E_6(\tau)E_{6,D_n}(\tau,z),\\
&E_8(\tau)E_{4,D_n}(\tau,z) - \frac{576}{(8-n)(12-n)}E_4(\tau)E_{8,D_n}(\tau,z) \text{ and}\\
&\frac{12-n}{24}E_6(\tau)E_{6,D_n}(\tau,z) + E_4(\tau)E_{8,D_n}(\tau,z)
\end{aligned}$$

are cusp forms in J_{12,D_n} when $n = 1, 5$ or 7. The matrix

$$\begin{pmatrix} \frac{8-n}{24} & 1 & 0\\ 1 & 0 & -\frac{576}{(8-n)(12-n)}\\ 0 & \frac{12-n}{24} & 1 \end{pmatrix}$$

has echelon form

$$\begin{pmatrix} 1 & 0 & -\frac{576}{(8-n)(12-n)}\\ 0 & 1 & \frac{24}{12-n}\\ 0 & 0 & 0 \end{pmatrix}.$$

In other words,

$$\begin{aligned}(3.24)\qquad &E_8(\tau)E_{4,D_n}(\tau,z)-\frac{576}{(8-n)(12-n)}E_4(\tau)E_{8,D_n}(\tau,z) \text{ and}\\ &E_6(\tau)E_{6,D_n}(\tau,z)+\frac{24}{12-n}E_4(\tau)E_{8,D_n}(\tau,z)\end{aligned}$$

form a basis of S_{12,D_n}. It follows from (3.19) that

$$\begin{aligned}&\frac{8-n}{24}E_{10}(\tau)E_{4,D_n}(\tau,z)+E_8(\tau)E_{6,D_n}(\tau,z),\\ &E_{10}(\tau)E_{4,D_n}(\tau,z)-\frac{576}{(8-n)(12-n)}E_6(\tau)E_{8,D_n}(\tau,z) \text{ and}\\ &\frac{12-n}{24}E_8(\tau)E_{6,D_n}(\tau,z)+E_6(\tau)E_{8,D_n}(\tau,z)\end{aligned}$$

are cusp forms in J_{14,D_n} when $n=1,5$ or 7 and, by the same reasoning as above,

$$\begin{aligned}(3.25)\qquad &E_{10}(\tau)E_{4,D_n}(\tau,z)-\frac{576}{(8-n)(12-n)}E_6(\tau)E_{8,D_n}(\tau,z) \text{ and}\\ &E_8(\tau)E_{6,D_n}(\tau,z)+\frac{24}{12-n}E_6(\tau)E_{8,D_n}(\tau,z)\end{aligned}$$

form a basis of S_{14,D_n}. Equation (3.17) implies that

$$\Delta(\tau)E_{t,D_n}(\tau,z)=\sum_{\substack{r\in D_n^{\#},D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}}\sum_{l=1}^{\lfloor -D\rfloor}C_{t,n}(D+l,r)\tau(l)e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau+(r,z)\right).$$

It follows from (3.19) that

$$\begin{aligned}&\Delta(\tau)E_{4,D_n}(\tau,z),\frac{8-n}{24}E_{12}(\tau)E_{4,D_n}(\tau,z)+E_{10}(\tau)E_{6,D_n}(\tau,z),\\ &E_{12}(\tau)E_{4,D_n}(\tau,z)-\frac{576}{(8-n)(12-n)}E_8(\tau)E_{8,D_n}(\tau,z) \text{ and}\\ &\frac{12-n}{24}E_{10}(\tau)E_{6,D_n}(\tau,z)+E_8(\tau)E_{8,D_n}(\tau,z)\end{aligned}$$

are cusp forms in J_{16,D_n} when $n=1,5$ or 7 and, by the same reasoning as above,

$$\begin{aligned}(3.26)\qquad &E_{12}(\tau)E_{4,D_n}(\tau,z)-\frac{576}{(8-n)(12-n)}E_8(\tau)E_{8,D_n}(\tau,z),\\ &E_{10}(\tau)E_{6,D_n}(\tau,z)+\frac{24}{12-n}E_8(\tau)E_{8,D_n}(\tau,z) \text{ and } \Delta(\tau)E_{4,n}(\tau,z)\end{aligned}$$

form a basis of S_{16,D_n}.

We computed the Hecke eigenvalues of Jacobi forms of index D_n using the fact that

$$T(l)\phi=\lambda_\phi(l)\phi$$

for every Hecke eigenform ϕ and by implementing the results from this section in SageMath [**The**]. Values of Fourier coefficients and Atkin–Lehner eigenvalues of elliptic newforms are available on the LMFDB web page [**The13**]. Let ϕ be an element of J_{k,D_n} ($n=1,3,5$ and 7). The eigenvalues of Jacobi forms of weights $4,6,8,10$ and 12 and index D_n were computed for odd positive integers l using the pair $(-1,(0,\dots,0))$ in the support of D_n. The eigenvalues of Jacobi forms of weights $12-n,16-n,18-n,20-n$ and $22-n$ and index D_n were computed for odd positive integers l using the pair $\left(-\frac{n-1}{8},\left(\frac{1}{2},\dots,\frac{1}{2}\right)\right)$ in the support of D_n, unless $(l,n-1)>1$. In the latter case, we replaced $-\frac{n-1}{8}$ with $-\frac{m}{8}$, where m is the smallest positive integer in the congruence class of $n-1$ modulo 8 which is coprime to l. For every odd, positive integer l and every negative integer D which is coprime to l, equation (3.7) implies that

$$C_{T(l)\phi}(D,(0,\dots,0)) = \sum_{d|l} d^{k-\lceil\frac{n}{2}\rceil-1}\left(\frac{(-1)^{\lfloor\frac{n}{2}\rfloor}8D}{d}\right)C_\phi\left(\frac{l^2}{d^2}D,(0,\dots,0)\right)$$

and for every negative rational number E such that $E \equiv \frac{n}{8} \bmod \mathbb{Z}$ and $(8E,l) = 1$, equation (3.7) implies that

$$C_{T(l)\phi}\left(E,\left(\frac{1}{2},\dots,\frac{1}{2}\right)\right) = \sum_{d|l} d^{k-\lceil\frac{n}{2}\rceil-1}\left(\frac{(-1)^{\lfloor\frac{n}{2}\rfloor}8E}{d}\right)C_\phi\left(\frac{l^2}{d^2}E,\frac{l}{d}\left(\frac{1}{2},\dots,\frac{1}{2}\right)\right).$$

3.3.2.1. *The lattice D_1.* It is straight-forward to check that $D_1 \simeq \underline{L}_2$. Theorem 1.37 implies that

$$J_{k,D_1} \simeq \mathfrak{M}^-_{2k-2}(2).$$

Since $M_t^\varepsilon(m)$ ($\varepsilon \in \{+,-\}$) denotes the subspace of $M_t(m)$ which is spanned by modular forms with eigenvalue εi^t with respect to Atkin–Lehner involutions, the space $\mathfrak{M}^-_{2k-2}(2)$ is spanned by modular forms with eigenvalue -1 with respect to Atkin–Lehner involutions when k is odd and by modular forms with eigenvalue $+1$ with respect to Atkin–Lehner involutions when k is even.

Equation (3.19) implies that $J_{4,D_1} = \mathbb{C}E_{4,D_1}$ and this space is mapped to $M_6(2)$. In particular, the Jacobi form E_{4,D_1} is a Hecke eigenform. We have checked that the first 13 Hecke eigenvalues of E_{4,D_1} at odd positive integers match the Fourier coefficients of $-E_6/504$.

Equation (3.19) implies that $J_{6,D_1} = \mathbb{C}E_{6,D_1}$ and this space is mapped to $M_{10}(2)$. In particular, the Jacobi form E_{6,D_1} is a Hecke eigenform. We have checked that the first 13 Hecke eigenvalues of E_{6,D_1} at odd positive integers match the Fourier coefficients of $-E_{10}/264$.

Equation (3.19) implies that

$$J_{8,D_1} = \mathbb{C}E_4E_{4,D_1} \oplus \mathbb{C}E_{8,D_1}$$

and this space is mapped to $M_{14}(2)$. Equation (3.22) implies that

$$\psi_{8,D_1}(\tau,z) := E_4(\tau)E_{4,D_1}(\tau,z) - \frac{576}{77}E_{8,D_1}(\tau,z)$$

is a Hecke eigenform in S_{8,D_1}. The first few Fourier coefficients of $11\psi_{8,D_1}$ are listed in Table A.1. The space $M_{14}(2)$ contains precisely two newforms:

$$\begin{aligned} f_{14}(\tau) =& q - 64q^2 - 1836q^3 + 4096q^4 + 3990q^5 + 117504q^6 - 433432q^7 - 262144q^8 \\ &+ 1776573q^9 - 255360q^{10} + 1619772q^{11} - 7520256q^{12} + O(q^{13}) \text{ and} \\ g_{14}(\tau) =& q + 64q^2 + 1236q^3 + 4096q^4 - 57450q^5 + 79104q^6 + 64232q^7 + 262144q^8 \\ &- 66627q^9 - 3676800q^{10} + 2464572q^{11} + 5062656q^{12} + O(q^{13}). \end{aligned}$$

The first 13 Hecke eigenvalues of ψ_{8,D_1} at odd positive integers match the Fourier coefficients of f_{14} and this newform is an element of $M^-_{14}(2)$.

Equation (3.19) implies that

$$J_{10,D_1} = \mathbb{C}E_6E_{4,D_1} \oplus \mathbb{C}E_4E_{6,D_1}$$

and this space is mapped to $M_{18}(2)$. Equation (3.23) implies that

$$\psi_{10,D_1}(\tau,z) := \frac{7}{24}E_6(\tau)E_{4,D_1}(\tau,z) + E_4(\tau)E_{6,D_1}(\tau,z)$$

is a Hecke eigenform in S_{10,D_1}. The space $M_{18}(2)$ contains precisely one newform:

$$\begin{aligned}f_{18}(\tau) =&q + 256q^2 + 6084q^3 + 65536q^4 + 1255110q^5 + 1557504q^6 - 22465912q^7\\&+ 16777216q^8 - 92125107q^9 + 321308160q^{10} + 172399692q^{11} + O(q^{12}),\end{aligned}$$

which is an element of $M^+_{18}(2)$. According to the LMFDB, the space $S_{18}(2)$ contains precisely two oldforms,

$$F_{18}(\tau) =q - 528q^2 - 4284q^3 + 147712q^4 - 1025850q^5 + 2261952q^6 + O(q^7)$$

and $F_{18}(2\tau)$. We have checked that the first 13 Hecke eigenvalues of ψ_{10,D_1} at odd positive integers match the Fourier coefficients of F_{18}.

Equation (3.18) implies that $J_{11,D_1} = \mathbb{C}\psi_{11,D_1}$ and this space is mapped to $M_{20}(2)$. In particular, the Jacobi form ψ_{11} is a Hecke eigenform. The space $M_{20}(2)$ contains precisely two newforms:

$$\begin{aligned}f_{20}(\tau) =&q + 512q^2 - 53028q^3 + 262144q^4 - 5556930q^5 - 27150336q^6 - 44496424q^7\\&+ 134217728q^8 + 1649707317q^9 - 2845148160q^{10} + O(q^{11}) \text{ and}\\ g_{20}(\tau) =&q - 512q^2 - 13092q^3 + 262144q^4 + 6546750q^5 + 6703104q^6 + O(q^7).\end{aligned}$$

We have checked that the first 13 Hecke eigenvalues of ψ_{11,D_1} at odd positive integers match the Fourier coefficients of f_{20} and this newform is an element of $M^-_{20}(2)$.

Equation (3.19) implies that

$$J_{12,D_1} = \mathbb{C}E_8E_{4,D_1} \oplus \mathbb{C}E_6E_{6,D_1} \oplus \mathbb{C}E_4E_{8,D_1}$$

and this space should be mapped to $M_{22}(2)$. Set

$$\begin{aligned}\beta^1_{12,1} :=&E_8E_{4,D_1} - \frac{576}{77}E_4E_{8,D_1} \text{ and}\\ \beta^2_{12,1} :=&E_6E_{6,D_1} + \frac{24}{11}E_4E_{8,D_1}.\end{aligned}$$

Equation (3.24) implies that these functions form a basis of S_{12,D_1}. The matrix of $T(l)$ on this space (which we denote by $T(l)$ as well by abuse of notation) satisfies

$$\begin{pmatrix}T(l)\beta^1_{12,1}\\ T(l)\beta^2_{12,1}\end{pmatrix} = T(l)\begin{pmatrix}\beta^1_{12,1}\\ \beta^2_{12,1}\end{pmatrix}.$$

Set $T(l) = \begin{pmatrix} a^l_{11} & a^l_{12}\\ a^l_{21} & a^l_{22}\end{pmatrix}$. We compute this matrix by solving the following two systems of linear equations:

$$\begin{pmatrix}a^l_{11}\\ a^l_{12}\end{pmatrix} = \begin{pmatrix}C_{\beta^1_{12,1}}(-1,0) & C_{\beta^2_{12,1}}(-1,0)\\ C_{\beta^1_{12,1}}(-2,0) & C_{\beta^2_{12,1}}(-2,0)\end{pmatrix}^{-1}\begin{pmatrix}C_{T(l)\beta^1_{12,1}}(-1,0)\\ C_{T(l)\beta^1_{12,1}}(-2,0)\end{pmatrix} \text{ and}$$

$$\begin{pmatrix}a^l_{21}\\ a^l_{22}\end{pmatrix} = \begin{pmatrix}C_{\beta^1_{12,1}}(-1,0) & C_{\beta^2_{12,1}}(-1,0)\\ C_{\beta^1_{12,1}}(-2,0) & C_{\beta^2_{12,1}}(-2,0)\end{pmatrix}^{-1}\begin{pmatrix}C_{T(l)\beta^2_{12,1}}(-1,0)\\ C_{T(l)\beta^2_{12,1}}(-2,0)\end{pmatrix}.$$

Using Sage, we obtain that

$$T(3) = \begin{pmatrix}-\frac{1458756}{11} & \frac{9953280}{77}\\ -\frac{65520}{11} & \frac{829116}{11}\end{pmatrix}$$

and this matrix can be diagonalized as

$$T(3) = \begin{pmatrix}1 & 1\\ \frac{455}{288} & \frac{7}{240}\end{pmatrix}\begin{pmatrix}71604 & 0\\ 0 & -128844\end{pmatrix}\begin{pmatrix}1 & 1\\ \frac{455}{288} & \frac{7}{240}\end{pmatrix}^{-1}.$$

It follows that the two Jacobi forms ψ_{12,D_1} and ϕ_{12,D_1} defined by the system of equations

$$\begin{pmatrix}\psi_{12,D_1}\\ \phi_{12,D_1}\end{pmatrix} := \begin{pmatrix}1 & 1\\ \frac{455}{288} & \frac{7}{240}\end{pmatrix}^{-1}\begin{pmatrix}\beta^1_{12,1}\\ \beta^2_{12,1}\end{pmatrix} = \begin{pmatrix}-\frac{6}{319}\beta^1_{12,1} + \frac{1440}{2233}\beta^2_{12,1}\\ \frac{325}{319}\beta^1_{12,1} - \frac{1440}{2233}\beta^2_{12,1}\end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{12,D_1}}(3) = 71604$ and $\lambda_{\phi_{12,D_1}}(3) = -128844$ respectively. The space $M_{22}(2)$ contains precisely two newforms:

$$\begin{aligned} f_{22}(\tau) =& q - 1024q^2 + 71604q^3 + 1048576q^4 - 28693770q^5 - 73322496q^6\\ &- 853202392q^7 - 1073741824q^8 - 5333220387q^9 + O(q^{10}) \text{ and}\\ g_{22}(\tau) =& q + 1024q^2 + 59316q^3 + 1048576q^4 + 4975350q^5 + 60739584q^6\\ &+ 1427425832q^7 + 1073741824q^8 - 6941965347q^9 + O(q^{10}). \end{aligned}$$

We have checked that the first 13 Hecke eigenvalues of ψ_{12,D_1} at odd positive integers match the Fourier coefficients of f_{22} and this newform is an element of $M^-_{22}(2)$. The space $S_{22}(2)$ contains precisely two oldforms,

$$F_{22}(\tau) = q - 288q^2 - 128844q^3 - 2014208q^4 + 21640950q^5 + 37107072q^6 + O(q^7)$$

and $F_{22}(2\tau)$. We have checked that the first 13 Hecke eigenvalues of ϕ_{12,D_1} at odd positive integers match the Fourier coefficients of F_{22}.

Equation (3.18) implies that $J_{13,D_1} = \{0\}$ and this space is mapped to $M_{24}(2)$. The latter space contains precisely one newform:

$$\begin{aligned} f_{24}(\tau) =& q - 2048q^2 - 505908q^3 + 4194304q^4 - 90135570q^5 + 1036099584q^6\\ &+ 6872255096q^7 - 8589934592q^8 + 161799725637q^9 + O(q^{10}), \end{aligned}$$

which is an element of $M^+_{24}(2)$. This agrees with the fact that $J_{13,D_1} \simeq \mathfrak{M}^-_{24}(2)$.

Equation (3.19) implies that

$$J_{14,D_1} = \mathbb{C}E_{10}E_{4,D_1} \oplus \mathbb{C}E_8E_{6,D_1} \oplus \mathbb{C}E_6E_{8,D_1}$$

and this space should be mapped to $M_{26}(2)$. Set

$$\begin{aligned} \beta^1_{14,1} :=& E_{10}E_{4,D_1} - \frac{576}{77}E_6E_{8,D_1} \text{ and}\\ \beta^2_{14,1} :=& E_8E_{6,D_1} + \frac{24}{11}E_6E_{8,D_1}. \end{aligned}$$

Equation (3.25) implies that these functions form a basis of S_{14,D_1}. Following the same argument as in the weight 12 case, the two Jacobi forms ψ_{14,D_1} and ϕ_{14,D_1} defined by the system of equations

$$\begin{pmatrix}\psi_{14,D_1}\\ \phi_{14,D_1}\end{pmatrix} := \begin{pmatrix}1 & 1\\ \frac{49}{162} & -\frac{181}{432}\end{pmatrix}^{-1}\begin{pmatrix}\beta^1_{14,1}\\ \beta^2_{14,1}\end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{14,D_1}}(3) = 97956$ and $\lambda_{\phi_{14,D_1}}(3) = -195804$ respectively. The space $M_{26}(2)$ contains precisely three newforms

$$\begin{aligned} f_{26}(\tau) =& q - 4096q^2 + 97956q^3 + 16777216q^4 + 341005350q^5 - 401227776q^6 + O(q^7),\\ g_{26}(\tau) =& q + 4096q^2 + (189924 - \beta)q^3 + 16777216q^4 + O(q^5) \text{ and}\\ \overline{g}_{26}(\tau) =& q + 4096q^2 + (189924 + \beta)q^3 + 16777216q^4 + O(q^5), \end{aligned}$$

where $\beta = 4800\sqrt{106705}$. We have checked that the first 13 eigenvalues of ψ_{14} at odd positive integers match the Fourier coefficients of f_{26} and this newform is an element of $M^-_{26}(2)$. The space $S_{26}(2)$ contains precisely two oldforms,

$$F_{26}(\tau) = q - 48q^2 - 195804q^3 - 33552128q^4 - 741989850q^5 + 9398592q^6 + O(q^7)$$

and $F_{26}(2\tau)$. We have checked that the first 13 Hecke eigenvalues of ϕ_{14,D_1} at odd positive integers match the Fourier coefficients of F_{26}.

Equation (3.18) implies that $J_{15,D_1} = \mathbb{C}E_4\psi_{11,D_1}$ and this space is mapped to $M_{28}(2)$. Set $\psi_{15,D_1} := E_4\psi_{11,D_1}$. In particular, this Jacobi form is a Hecke eigenform. The space $M_{28}(2)$ contains precisely two newforms:

$$\begin{aligned} f_{28}(\tau) =& q + 8192q^2 - 1016388q^3 + 67108864q^4 - 3341197410q^5 - 8326250496q^6 \\ &- 51021361384q^7 + 549755813888q^8 - 6592552918443q^9 + O(q^{10}) \text{ and} \\ g_{28}(\tau) =& q - 8192q^2 + 3984828q^3 + 67108864q^4 - 2851889250q^5 - 32643710976q^6 \\ &+ 368721063704q^7 - 549755813888q^8 + 8253256704597q^9 + O(q^{10}). \end{aligned}$$

We have checked that the first 10 Hecke eigenvalues of ψ_{15,D_1} at odd positive integers match the Fourier coefficients of f_{28} and this newform is an element of $M^-_{28}(2)$.

Equation (3.19) implies that

$$J_{16,D_1} = \mathbb{C}\Delta E_{4,D_1} \oplus \mathbb{C}E_{12}E_{4,D_1} \oplus \mathbb{C}E_{10}E_{6,D_1} \oplus \mathbb{C}E_8E_{8,D_1}$$

and this space should be mapped to $M_{30}(2)$. Set

$$\begin{aligned} \beta^1_{16,1} :=& E_{12}E_{4,D_1} - \frac{576}{77}E_8E_{8,D_1}, \\ \beta^2_{16,1} :=& E_{10}E_{6,D_1} + \frac{24}{11}E_8E_{8,D_1} \text{ and} \\ \beta^3_{16,1} :=& \Delta E_{4,D_1}. \end{aligned}$$

Equation (3.26) implies that these functions form a basis of S_{16,D_1}. The matrix of $T(l)$ on this space satisfies

$$\begin{pmatrix} T(l)\beta^1_{16,1} \\ T(l)\beta^2_{16,1} \\ T(l)\beta^3_{16,1} \end{pmatrix} = T(l)\begin{pmatrix} \beta^1_{16,1} \\ \beta^2_{16,1} \\ \beta^3_{16,1} \end{pmatrix}.$$

Set

$$T(l) = \begin{pmatrix} a^l_{11} & a^l_{12} & a^l_{13} \\ a^l_{21} & a^l_{22} & a^l_{23} \\ a^l_{31} & a^l_{32} & a^l_{33} \end{pmatrix}.$$

We compute this matrix by solving the following three systems of linear equations:

$$\begin{pmatrix} a^l_{11} \\ a^l_{12} \\ a^l_{13} \end{pmatrix} = \begin{pmatrix} C_{\beta^1_{16,1}}(-1,0) & C_{\beta^2_{16,1}}(-1,0) & C_{\beta^3_{16,1}}(-1,0) \\ C_{\beta^1_{16,1}}(-2,0) & C_{\beta^2_{16,1}}(-2,0) & C_{\beta^3_{16,1}}(-2,0) \\ C_{\beta^1_{16,1}}(-4,0) & C_{\beta^2_{16,1}}(-4,0) & C_{\beta^3_{16,1}}(-4,0) \end{pmatrix}^{-1} \begin{pmatrix} C_{T(l)\beta^1_{16,1}}(-1,0) \\ C_{T(l)\beta^1_{16,1}}(-2,0) \\ C_{T(l)\beta^1_{16,1}}(-4,0) \end{pmatrix},$$

$$\begin{pmatrix} a^l_{21} \\ a^l_{22} \\ a^l_{23} \end{pmatrix} = \begin{pmatrix} C_{\beta^1_{16,1}}(-1,0) & C_{\beta^2_{16,1}}(-1,0) & C_{\beta^3_{16,1}}(-1,0) \\ C_{\beta^1_{16,1}}(-2,0) & C_{\beta^2_{16,1}}(-2,0) & C_{\beta^3_{16,1}}(-2,0) \\ C_{\beta^1_{16,1}}(-4,0) & C_{\beta^2_{16,1}}(-4,0) & C_{\beta^3_{16,1}}(-4,0) \end{pmatrix}^{-1} \begin{pmatrix} C_{T(l)\beta^2_{16,1}}(-1,0) \\ C_{T(l)\beta^2_{16,1}}(-2,0) \\ C_{T(l)\beta^2_{16,1}}(-4,0) \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} a^l_{31} \\ a^l_{32} \\ a^l_{33} \end{pmatrix} = \begin{pmatrix} C_{\beta^1_{16,1}}(-1,0) & C_{\beta^2_{16,1}}(-1,0) & C_{\beta^3_{16,1}}(-1,0) \\ C_{\beta^1_{16,1}}(-2,0) & C_{\beta^2_{16,1}}(-2,0) & C_{\beta^3_{16,1}}(-2,0) \\ C_{\beta^1_{16,1}}(-4,0) & C_{\beta^2_{16,1}}(-4,0) & C_{\beta^3_{16,1}}(-4,0) \end{pmatrix}^{-1} \begin{pmatrix} C_{T(l)\beta^3_{16,1}}(-1,0) \\ C_{T(l)\beta^3_{16,1}}(-2,0) \\ C_{T(l)\beta^3_{16,1}}(-4,0) \end{pmatrix}.$$

Using Sage, we obtain that

$$T(3) = \begin{pmatrix} \frac{27196229844}{7601} & -\frac{1516424509440}{53207} & \frac{8998059438489600}{477481} \\ \frac{45758160}{11} & -\frac{215457444}{11} & \frac{8874947450880}{691} \\ 2625 & -\frac{4848}{7} & \frac{5699964924}{691} \end{pmatrix}$$

and this matrix can be diagonalized as

$$T(3) = \begin{pmatrix} 1 & 1 & 1 \\ \frac{-179786453\sqrt{51349}+207180739619}{292431960720} & \frac{179786453\sqrt{51349}+207180739619}{292431960720} & \frac{24185}{349512} \\ \frac{-5252291\sqrt{51349}}{1403673411456}+\frac{328902871}{438647941080} & \frac{5252291\sqrt{51349}}{1403673411456}+\frac{328902871}{438647941080} & -\frac{11747}{50329728} \end{pmatrix}$$
$$\times \begin{pmatrix} -52992\sqrt{51349}-2483820 & 0 & 0 \\ 0 & 52992\sqrt{51349}-2483820 & 0 \\ 0 & 0 & -2792556 \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 1 & 1 \\ \frac{-179786453\sqrt{51349}+207180739619}{292431960720} & \frac{179786453\sqrt{51349}+207180739619}{292431960720} & \frac{24185}{349512} \\ \frac{-5252291\sqrt{51349}}{1403673411456}+\frac{328902871}{438647941080} & \frac{5252291\sqrt{51349}}{1403673411456}+\frac{328902871}{438647941080} & -\frac{11747}{50329728} \end{pmatrix}^{-1}.$$

It follows that the three Jacobi forms ψ_{16,D_1}, ϕ_{16,D_1} and δ_{16,D_1} defined by the system of equations

$$\begin{pmatrix} \psi_{16,D_1} \\ \phi_{16,D_1} \\ \delta_{16,D_1} \end{pmatrix} := \begin{pmatrix} 1 & 1 & 1 \\ \frac{-179786453\sqrt{51349}+207180739619}{292431960720} & \frac{179786453\sqrt{51349}+207180739619}{292431960720} & \frac{24185}{349512} \\ \frac{-5252291\sqrt{51349}}{1403673411456}+\frac{328902871}{438647941080} & \frac{5252291\sqrt{51349}}{1403673411456}+\frac{328902871}{438647941080} & -\frac{11747}{50329728} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{16,1} \\ \beta^2_{16,1} \\ \beta^3_{16,1} \end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{16,D_1}}(3) = -52992\sqrt{51349}-2483820$, $\lambda_{\phi_{16,D_1}}(3) = 52992\sqrt{51349} - 2483820$ and $\lambda_{\delta_{16,D_1}}(3) = -2792556$, respectively. The space $M_{30}(2)$ contains precisely two newforms:

$$\begin{aligned} f_{30}(\tau) =& q - 16384q^2 - 2792556q^3 + 268435456q^4 + 6651856470q^5 \\ &+ 45753237504q^6 + 1432518476648q^7 - 4398046511104q^8 + O(q^9) \text{ and} \\ g_{30}(\tau) =& q + 16384q^2 + 4782996q^3 + 268435456q^4 + 6065841750q^5 + O(q^6). \end{aligned}$$

We have checked that the first 13 Hecke eigenvalues of δ_{16,D_1} at odd positive integers match the Fourier coefficients of f_{30} and this newform is an element of $M^-_{30}(2)$. The space $S_{30}(2)$ contains precisely four oldforms,

$$\begin{aligned} F_{30}(\tau) =& q + (4320 - \beta)q^2 + (-2483820 + 552\beta)q^3 + (-44976128 - 8640\beta)q^4 \\ &+ (-8738894250 - 116000\beta)q^5 + (-271954378368 + 4868460\beta)q^6 + O(q^7), \\ \overline{F}_{30}(\tau) =& q + (4320 + \beta)q^2 + (-2483820 - 552\beta)q^3 + (-44976128 + 8640\beta)q^4 + O(q^5), \end{aligned}$$

$F_{30}(2\tau)$ and $\overline{F}_{30}(2\tau)$, where $\beta = 96\sqrt{51349}$. We have checked that the first 13 Hecke eigenvalues of ϕ_{16,D_1} and ψ_{16,D_1} at odd positive integers match the Fourier coefficients of F_{30} and $\overline{F}_{30}$, respectively.

Equation (3.18) implies that $J_{17,D_1} = \mathbb{C}E_6\psi_{11,D_1}$ and this space is mapped to $M_{32}(2)$. Set $\psi_{17,D_1} := E_6\psi_{11,D_1}$. In particular, this Jacobi form is a Hecke eigenform. The space $M_{32}(2)$ contains precisely three newforms:

$$\begin{aligned} f_{32}(\tau) =& q + 32768q^2 - 19984212q^3 + 1073741824q^4 + 42951708750q^5 \\ &- 654842658816q^6 - 16835358997576q^7 + 35184372088832q^8 + O(q^9), \\ g_{32}(\tau) =& q - 32768q^2 + (8358252 - \beta)q^3 + 1073741824q^4 + O(q^5) \text{ and} \\ \overline{g}_{32}(\tau) =& q - 32768q^2 + (8358252 + \beta)q^3 + 1073741824q^4 + O(q^5), \end{aligned}$$

where $\beta = 960\sqrt{987507049}$. We have checked that the first 10 Hecke eigenvalues of ψ_{17,D_1} at odd positive integers match the Fourier coefficients of f_{32} and this newform is an element of $M^-_{32}(2)$.

Equation (3.18) implies that $J_{19,D_1} = \mathbb{C}E_8\psi_{11,D_1}$ and this space is mapped to $M_{36}(2)$. Set $\psi_{19,D_1} := E_8\psi_{11,D_1}$. In particular, this Jacobi form is a Hecke eigenform. The space

$M_{36}(2)$ contains precisely two newforms

$$\begin{aligned}
f_{36}(\tau) =& q + 131072q^2 + 159933852q^3 + 17179869184q^4 - 2838742578690q^5 \\
&+ 20962849849344q^6 - 782281866962344q^7 + O(q^8) \text{ and} \\
g_{36}(\tau) =& q - 131072q^2 + 36494748q^3 + 17179869184q^4 + 389070858750q^5 \\
&- 4783439609856q^6 - 129689369490856q^7 + O(q^8).
\end{aligned}$$

We have checked that the first 10 Hecke eigenvalues of ψ_{19,D_1} at odd positive integers match the Fourier coefficients of f_{36} and this newform is an element of $M_{36}^-(2)$.

Equation (3.18) implies that $J_{21,D_1} = \mathbb{C}E_{10}\psi_{11,D_1}$ and this space is mapped to $M_{40}(2)$. Set $\psi_{21,D_1} := E_{10}\psi_{11,D_1}$. In particular, this Jacobi form is a Hecke eigenform. The space $M_{40}(2)$ contains precisely three newforms:

$$\begin{aligned}
f_{40}(\tau) =& q + 524288q^2 - 735458292q^3 + 274877906944q^4 - 16226178983250q^5 \\
&- 385591956996096q^6 + 16050065775887864q^7 + O(q^8), \\
g_{40}(\tau) =& q - 524288q^2 + (143709132 - \beta)q^3 + O(q^5), \\
\overline{g}_{40}(\tau) =& q - 524288q^2 + (143709132 + \beta)q^3 + 274877906944q^4 + O(q^5),
\end{aligned}$$

where $\beta = 960\sqrt{4202094647521}$. We have checked that the first 10 Hecke eigenvalues of ψ_{21,D_1} at odd positive integers match the Fourier coefficients of f_{40} and this newform is an element of $M_{40}^-(2)$.

Equation (3.18) implies that

$$J_{23,D_1} = S_{23,1} = \mathbb{C}E_{12}\psi_{11,D_1} \oplus \mathbb{C}\Delta\psi_{11,D_1}$$

and this space should be mapped to $M_{44}(2)$. Set

$$\begin{aligned}
\beta^1_{23,1} :=& E_{12}\psi_{11,D_1} \text{ and} \\
\beta^2_{23,1} :=& \Delta\psi_{11,D_1}.
\end{aligned}$$

The matrix of $T(l)$ on this basis satisfies

$$\begin{pmatrix} T(l)\beta^1_{23,1} \\ T(l)\beta^2_{23,1} \end{pmatrix} = T(l)\begin{pmatrix} \beta^1_{23,1} \\ \beta^2_{23,1} \end{pmatrix}$$

and it can be computed following the same reasoning as before. Consequently, the two Jacobi forms ψ_{23,D_1} and ϕ_{23,D_1} defined by the system of equations

$$\begin{pmatrix} \psi_{23,D_1} \\ \phi_{23,D_1} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{477481\sqrt{1589985537001}}{2659891394160000} - \frac{192561338303}{1329945697080000} & -\frac{477481\sqrt{1589985537001}}{2659891394160000} - \frac{192561338303}{1329945697080000} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{23,1} \\ \beta^2_{23,1} \end{pmatrix}$$

are Hecke eigenforms. The space $M_{44}(2)$ contains precisely four newforms:

$$\begin{aligned}
f_{44}(\tau) =& q + 2097152q^2 + (-11170817028 - \alpha)q^3 + 4398046511104q^4 + O(q^5), \\
\overline{f}_{44}(\tau) =& q + 2097152q^2 + (-11170817028 + \alpha)q^3 + 4398046511104q^4 + O(q^5), \\
g_{44}(\tau) =& q - 2097152q^2 + (-6490815492 - \beta)q^3 + 4398046511104q^4 + O(q^5) \text{ and} \\
\overline{g}_{44}(\tau) =& q - 2097152q^2 + (-6490815492 + \beta)q^3 + 4398046511104q^4 + O(q^5),
\end{aligned}$$

where $\alpha = 17280\sqrt{1589985537001}$ and $\beta = 21120\sqrt{97578078049}$. We have checked that the first 10 Hecke eigenvalues of ψ_{23,D_1} and ϕ_{23,D_1} at odd positive integers match the Fourier coefficients of f_{44} and $\overline{f}_{44}$, respectively, and these newforms are elements of $M_{44}^-(2)$.

3.3.2.2. *The lattice D_3.* For the lattice D_3, Remarks 3.12 and 3.16 suggest that there exists a lifting map

$$J_{k,D_3} \to M^-_{2k-4}(2).$$

Note that $M^-_{2k-4}(2)$ is spanned by modular forms with eigenvalue -1 with respect to Atkin–Lehner involutions when k is even and by modular forms with eigenvalue $+1$ with respect to Atkin–Lehner involutions when k is odd.

Equation (3.20) implies that $J_{4,D_3} = \mathbb{C}E_{4,D_3}$ and this space should be mapped to $M_4(2)$. In particular, the Jacobi form E_{4,D_3} is a Hecke eigenform. We have checked that the first 25 Hecke eigenvalues of E_{4,D_3} at odd positive integers match the Fourier coefficients of $E_4/240$.

Equation (3.20) implies that $J_{6,D_3} = \mathbb{C}E_{6,D_3}$ and this space should be mapped to $M_8(2)$. In particular, the Jacobi form E_{6,D_3} is a Hecke eigenform. We have checked that the first 25 Hecke eigenvalues of E_{6,D_3} at odd positive integers match the Fourier coefficients of $E_8/480$.

Equation (3.20) implies that

$$J_{8,D_3} = \mathbb{C}E_4E_{4,D_3} \oplus \mathbb{C}\eta^{12}\sigma_3^*\vartheta_{\mathbb{Z}^4}$$

and this space should be mapped to $M_{12}(2)$. Set $\psi_{8,D_3} := \eta^{12}\sigma_3^*\vartheta_{\mathbb{Z}^4}$. In particular, this Jacobi form is a Hecke eigenform in S_{8,D_3}. The first few Fourier coefficients of ψ_{8,D_3} are listed in Table A.2. The space $M_{12}(2)$ contains no newforms. The space $S_{12}(2)$ contains precisely two oldforms, $\Delta(\tau)$ and $\Delta(2\tau)$. The first 25 Hecke eigenvalues of ψ_{8,D_3} at odd positive integers match the Fourier coefficients of Δ.

Equation (3.18) implies that $J_{9,D_3} = \mathbb{C}\psi_{9,D_3}$ and this space should be mapped to $M_{14}(2)$. In particular, the Jacobi form ψ_{9,D_3} is a Hecke eigenform. The first few Fourier coefficients of ψ_{9,D_3} are listed in the second column of Table A.3. We have seen that $M_{14}(2)$ contains precisely two newforms and the first 25 Hecke eigenvalues of ψ_{9,D_3} at odd positive integers match the Fourier coefficients of f_{14}.

Equation (3.20) implies that

$$J_{10,D_3} = \mathbb{C}E_6E_{4,D_3} \oplus \mathbb{C}E_4E_{6,D_3}$$

and this space should be mapped to $M_{16}(2)$. Equation (3.23) implies that

$$\psi_{10,D_3}(\tau,z) := \frac{5}{24}E_6(\tau)E_{4,D_3}(\tau,z) + E_4(\tau)E_{6,D_3}(\tau,z)$$

is a Hecke eigenform in S_{10,D_3}. The space $M_{16}(2)$ contains precisely one newform:

$$\begin{aligned} f_{16}(\tau) =& q - 128q^2 + 6252q^3 + 16384q^4 + 90510q^5 - 800256q^6 + 56q^7 \\ &- 2097152q^8 + 24738597q^9 - 11585280q^{10} - 95889948q^{11} + O(q^{12}), \end{aligned}$$

which is an element of $M^+_{16}(2)$. The space $S_{16}(2)$ contains precisely two oldforms,

$$F_{16}(\tau) = q + 216q^2 - 3348q^3 + 13888q^4 + 52110q^5 - 723168q^6 + 2822456q^7 + O(q^8)$$

and $F_{16}(2\tau)$. We have checked that the first 25 Hecke eigenvalues of ψ_{10,D_3} at odd positive integers match the Fourier coefficients of F_{16}.

Equation (3.18) implies that $J_{11,D_3} = \{0\}$ and this space should be mapped to $M_{18}(2)$. We have seen that the latter space contains precisely one newform, which is an element of $M^+_{18}(2)$.

Equation (3.20) implies that

$$J_{12,D_3} = \mathbb{C}E_8E_{4,D_3} \oplus \mathbb{C}E_6E_{6,D_3} \oplus \mathbb{C}E_4\eta^{12}\sigma_3^*\vartheta_{\mathbb{Z}^4}$$

and this space should be mapped to $M_{20}(2)$. It follows that

$$\begin{aligned}\beta^1_{12,3}(\tau,z) :=& \frac{5}{24}E_8(\tau)E_{4,D_3}(\tau,z) + E_6(\tau)E_{6,D_3}(\tau,z)\\ =& \sum_{\substack{r\in D_3^\#, D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \Bigg[\frac{5}{24}C_{4,3}(D,r) + C_{6,3}(D,r) + \sum_{n=1}^{\lfloor -D\rfloor}\Big(100C_{4,3}(D+n,r)\sigma_7(n)\\ &- 504C_{6,3}(D+n,r)\sigma_5(n)\Big)\Bigg] e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau + (r,z)\right)\end{aligned}$$

is an element of S_{12,D_3}. On the other hand, set $\beta^2_{12,3} := E_4\eta^{12}\sigma_3^*\vartheta_{\mathbb{Z}^4}$. Equation (3.17) implies that

$$\begin{aligned}E_t(\tau)\eta^{12}\sigma_3^*\vartheta_{\mathbb{Z}^4}(\tau,z) =& \sum_{\substack{r\in D_3^\#, D\in\mathbb{Q}_{<0}\\ \frac{(r,r)}{2}-D\in\mathbb{Z}}} \Big(C_{\psi_{8,D_3}}(D,r) - \frac{2t}{B_t}\sum_{n=1}^{\lfloor -D\rfloor} C_{\psi_{8,D_3}}(D+n,r)\sigma_{t-1}(n)\Big)\\ &\times e\left(\left(\tfrac{(r,r)}{2}-D\right)\tau + (r,z)\right).\end{aligned}$$

The Jacobi forms $\beta^1_{12,3}$ and $\beta^2_{12,3}$ form a basis of S_{12,D_3} and the matrix of $T(l)$ on this space satisfies

$$\begin{pmatrix} T(l)\beta^1_{12,3}\\ T(l)\beta^2_{12,3}\end{pmatrix} = T(l)\begin{pmatrix}\beta^1_{12,3}\\ \beta^2_{12,3}\end{pmatrix}.$$

Using Sage, we obtain that

$$T(3) = \begin{pmatrix} -29988 & 403200\\ 4608 & 27612\end{pmatrix}$$

and this matrix can be diagonalized as

$$T(3) = \begin{pmatrix}1 & 1\\ \frac{1}{5} & -\frac{2}{35}\end{pmatrix}\begin{pmatrix}50652 & 0\\ 0 & -53028\end{pmatrix}\begin{pmatrix}1 & 1\\ \frac{1}{5} & -\frac{2}{35}\end{pmatrix}^{-1}.$$

It follows that the two Jacobi forms ψ_{12,D_3} and ϕ_{12,D_3} defined by the system of equations

$$\begin{pmatrix}\psi_{12,D_3}\\ \phi_{12,D_3}\end{pmatrix} := \begin{pmatrix}1 & 1\\ \frac{1}{5} & -\frac{2}{35}\end{pmatrix}^{-1}\begin{pmatrix}\beta^1_{12,3}\\ \beta^2_{12,3}\end{pmatrix} = \begin{pmatrix}\frac{2}{9}\beta^1_{12,3} + \frac{35}{9}\beta^2_{12,3}\\ \frac{7}{9}\beta^1_{12,3} - \frac{35}{9}\beta^2_{12,3}\end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{12,D_3}}(3) = 50652$ and $\lambda_{\phi_{12,D_3}}(3) = -53028$, respectively. We have seen that $M_{20}(2)$ contains precisely two newforms and we have checked that the first few Hecke eigenvalues of ϕ_{12,D_3} match the Fourier coefficients of f_{20} at odd integers. The space $S_{20}(2)$ contains precisely two oldforms,

$$F_{20}(\tau) = q + 456q^2 + 50652q^3 - 316352q^4 - 2377410q^5 + 23097312q^6 + O(q^7)$$

and $F_{20}(2\tau)$. We have checked that the first 19 Hecke eigenvalues of ψ_{12,D_3} at odd positive integers match the Fourier coefficients of F_{20}.

Equation (3.18) implies that $J_{13,D_3} = \mathbb{C}E_4\psi_{9,D_3}$ and this space should be mapped to $M_{22}(2)$. Set $\psi_{13,D_3} := E_4\psi_{9,D_3}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that $M_{22}(2)$ contains precisely two newforms and we have checked that the first 19 Hecke eigenvalues of ψ_{13,D_3} at odd positive integers match the Fourier coefficients of f_{22}.

Equation (3.20) implies that

$$J_{14,D_3} = \mathbb{C}E_{10}E_{4,D_3} \oplus \mathbb{C}E_8E_{6,D_3} \oplus \mathbb{C}E_6\psi_{8,D_3}$$

and this space should be mapped to $M_{24}(2)$. Set

$$\beta^1_{14,3}(\tau,z) := \frac{5}{24}E_{10}(\tau)E_{4,D_3}(\tau,z) + E_8(\tau)E_{6,D_3}(\tau,z)$$

and $\beta^2_{14,3} := E_6\psi_{8,D_3}$. Equation (3.17) implies that $\beta^1_{14,3}$ and $\beta^2_{14,3}$ form a basis of S_{14,D_3}. Following the same argument as in the weight 12 case, the two Jacobi forms ψ_{14,D_3} and ϕ_{14,D_3} defined by the system of equations

$$\begin{pmatrix}\psi_{14,D_3}\\ \phi_{14,D_3}\end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{-\sqrt{144169}-247}{1100} & \frac{\sqrt{144169}-247}{1100}\end{pmatrix}^{-1}\begin{pmatrix}\beta^1_{14,3}\\ \beta^2_{14,3}\end{pmatrix}$$

are Hecke eigenforms of $T(3)$. We have seen that $M_{24}(2)$ contains precisely one newform. The space $S_{24}(2)$ contains precisely four oldforms,

$$\begin{aligned} F_{24}(\tau) =&q + (540-\beta)q^2 + (169740+48\beta)q^3 + (12663328 - 1080\beta)q^4\\ &+ (36534510 + 15040\beta)q^5 + (-904836528 - 143820\beta)q^6\\ &+ (-679592200 + 985824\beta)q^7 + (24729511680 - 4857920\beta)q^8\\ &+ (-17499697083 + 16295040\beta)q^9 + O(q^{10}),\\ \overline{F}_{24}(\tau) =&q + (540+\beta)q^2 + (169740-48\beta)q^3 + (12663328 + 1080\beta)q^4 + O(q^5),\end{aligned}$$

$F_{24}(2\tau)$ and $\overline{F}_{24}(2\tau)$, where $\beta = 12\sqrt{144169}$. We have checked that the first 19 Hecke eigenvalues of ψ_{14,D_3} and ϕ_{14,D_3} at odd positive integers match the Fourier coefficients of F_{24} and $\overline{F}_{24}$, respectively.

Equation (3.18) implies that $J_{15,D_3} = \mathbb{C}E_6\psi_{9,D_3}$ and this space should be mapped to $M_{26}(2)$. Set $\psi_{15,D_3} := E_6\psi_{9,D_3}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that the space $M_{26}(2)$ contains precisely three newforms and we have checked that the first 19 Hecke eigenvalues of ψ_{15,D_3} at odd positive integers match the Fourier coefficients of f_{26}.

Equation (3.20) implies that

$$J_{16,D_3} = \mathbb{C}\Delta E_{4,D_3} \oplus \mathbb{C}E_{12}E_{4,D_3} \oplus \mathbb{C}E_{10}E_{6,D_3} \oplus \mathbb{C}E_8\psi_{8,D_3}$$

and this space should be mapped to $M_{28}(2)$. Set

$$\beta^1_{16,3}(\tau,z) := \frac{5}{24}E_{12}(\tau)E_{4,D_3}(\tau,z) + E_{10}(\tau)E_{6,D_3}(\tau,z),$$

$\beta^2_{16,3} := E_8\psi_{8,D_3}$ and $\beta^3_{16,3} := \Delta E_{4,D_3}$. Equation (3.17) implies that these functions form a basis of S_{16,D_3}. Using Sage, we obtain that

$$T(3) = \begin{pmatrix} \frac{1127316852}{691} & \frac{2874009600}{691} & -\frac{36453937766400}{477481}\\ 642816 & -2882628 & \frac{6270566400}{691}\\ -\frac{816}{5} & 3888 & -\frac{726564492}{691}\end{pmatrix}$$

on this basis and this matrix can be diagonalized as

$$\begin{aligned} T(3) =& \begin{pmatrix} 1 & 1 & 1\\ \frac{-76701\sqrt{18209}-8332078}{15791545} & \frac{76701\sqrt{18209}-8332078}{15791545} & \frac{691}{990}\\ \frac{477481\sqrt{18209}+71530247}{68219474400} & \frac{-477481\sqrt{18209}+71530247}{68219474400} & \frac{691}{9504}\end{pmatrix}\\ &\times\begin{pmatrix} -20736\sqrt{18209}-643140 & 0 & 0\\ 0 & 20736\sqrt{18209}-643140 & 0\\ 0 & 0 & -1016388\end{pmatrix}\\ &\times\begin{pmatrix} 1 & 1 & 1\\ \frac{-76701\sqrt{18209}-8332078}{15791545} & \frac{76701\sqrt{18209}-8332078}{15791545} & \frac{691}{990}\\ \frac{477481\sqrt{18209}+71530247}{68219474400} & \frac{-477481\sqrt{18209}+71530247}{68219474400} & \frac{691}{9504}\end{pmatrix}^{-1}.\end{aligned}$$

It follows that the three Jacobi forms ψ_{16,D_3}, ϕ_{16,D_3} and δ_{16,D_3} defined by the system of equations

$$\begin{pmatrix} \psi_{16,D_3} \\ \phi_{16,D_3} \\ \delta_{16,D_3} \end{pmatrix} := \begin{pmatrix} 1 & 1 & 1 \\ \frac{-76701\sqrt{18209}-8332078}{15791545} & \frac{76701\sqrt{18209}-8332078}{15791545} & \frac{691}{990} \\ \frac{477481\sqrt{18209}+71530247}{68219474400} & \frac{-477481\sqrt{18209}+71530247}{68219474400} & \frac{691}{9504} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{16,3} \\ \beta^2_{16,3} \\ \beta^3_{16,3} \end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{16,D_3}}(3) = -20736\sqrt{18209} - 643140$, $\lambda_{\phi_{16,D_3}}(3) = 20736\sqrt{18209} - 643140$ and $\lambda_{\delta_{16,D_3}}(l) = -1016388$, respectively. We have seen that $M_{28}(2)$ contains precisely two newforms and we have checked that the first 19 Hecke eigenvalues of δ_{16,D_3} at odd positive integers match the Fourier coefficients of f_{28}. The space $S_{28}(2)$ contains precisely four oldforms,

$$\begin{aligned} F_{28}(\tau) =& q + (-4140 - \beta)q^2 + (-643140 - 192\beta)q^3 + (95311648 + 8280\beta)q^4 \\ &+ (2721793950 - 147200\beta)q^5 + (43441436592 + 1438020\beta)q^6 + O(q^7), \\ \overline{F}_{28}(\tau) =& q + (-4140 + \beta)q^2 + (-643140 + 192\beta)q^3 + (95311648 - 8280\beta)q^4 + O(q^5), \end{aligned}$$

$F_{28}(2\tau)$ and $\overline{F}_{28}(2\tau)$, where $\beta = 108\sqrt{18209}$. We have checked that the first 19 Hecke eigenvalues of ψ_{16,D_3} and ϕ_{16,D_3} at odd positive integers match the Fourier coefficients of F_{28} and $\overline{F}_{28}$, respectively.

Equation (3.18) implies that $J_{17,D_3} = \mathbb{C}E_8\psi_{9,D_3}$ and this space should be mapped to $M_{30}(2)$. Set $\psi_{17,D_3} := E_8\psi_{9,D_3}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that $M_{30}(2)$ contains precisely two newforms and we have checked that the first 19 Hecke eigenvalues of ψ_{17,D_3} at odd positive integers match the Fourier coefficients of f_{30}.

Equation (3.18) implies that $J_{19,D_3} = \mathbb{C}E_{10}\psi_{9,D_3}$ and this space should be mapped to $M_{34}(2)$. Set $\psi_{19,D_3} := E_{10}\psi_{9,D_3}$. In particular, this Jacobi form is a Hecke eigenform. The space $M_{34}(2)$ contains precisely three newforms:

$$\begin{aligned} f_{34}(\tau) =& q - 65536q^2 - 133005564q^3 + 4294967296q^4 + 538799132550q^5 \\ &+ 8716652642304q^6 - 33347311051768q^7 - 281474976710656q^8 + O(q^9), \\ g_{34}(\tau) =& q + 65536q^2 + (4178244 - \beta)q^3 + 4294967296q^4 + (-2666238330 - 3996\beta)q^5 \\ &+ (273825398784 - 65536\beta)q^6 + (66359547937928 + 896238\beta)q^7 + O(q^8), \\ \overline{g}_{34}(\tau) =& q + 65536q^2 + (4178244 + \beta)q^3 + 4294967296q^4 + O(q^5), \end{aligned}$$

where $\beta = 10560\sqrt{79829689}$. We have checked that the first 19 Hecke eigenvalues of ψ_{19,D_3} at odd positive integers match the Fourier coefficients of f_{34} and this newform is an element of $M^-_{34}(2)$.

Equation (3.18) implies that

$$J_{21,D_3} = S_{21,3} = \mathbb{C}E_{12}\psi_{9,D_3} \oplus \mathbb{C}\Delta\psi_{9,D_3}$$

and this space should be mapped to $M_{38}(2)$. Set

$$\begin{aligned} \beta^1_{21,3} :=& E_{12}\psi_{9,D_3} \text{ and} \\ \beta^2_{21,3} :=& \Delta\psi_{9,D_3}. \end{aligned}$$

The matrix of $T(l)$ on this basis satisfies

$$\begin{pmatrix} T(l)\beta^1_{21,3} \\ T(l)\beta^2_{21,3} \end{pmatrix} = T(l) \begin{pmatrix} \beta^1_{21,3} \\ \beta^2_{21,3} \end{pmatrix}$$

and it can be computed using the same reasoning as before. Consequently, the two Jacobi forms ψ_{21,D_3} and ϕ_{21,D_3} defined by the system of equations

$$\begin{pmatrix}\psi_{21,D_3}\\ \phi_{21,D_3}\end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{477481\sqrt{3026574721}}{188339617065600} - \frac{10009989767}{94169808532800} & -\frac{477481\sqrt{3026574721}}{188339617065600} - \frac{10009989767}{94169808532800}\end{pmatrix}^{-1} \begin{pmatrix}\beta^1_{21,3}\\ \beta^2_{21,3}\end{pmatrix}$$

are Hecke eigenforms. The space $M_{38}(2)$ contains precisely four newforms:

$$\begin{aligned}
f_{38}(\tau) &= q - 262144q^2 + (211535604 - \alpha)q^3 + 68719476736q^4 + O(q^5),\\
\overline{f}_{38}(\tau) &= q - 262144q^2 + (211535604 + \alpha)q^3 + 68719476736q^4 + O(q^5),\\
g_{38}(\tau) &= q + 262144q^2 + (-250843404 - \beta)q^3 + 68719476736q^4 + O(q^5) \text{ and}\\
\overline{g}_{38}(\tau) &= q + 262144q^2 + (-250843404 - \beta)q^3 + 68719476736q^4 + O(q^5),
\end{aligned}$$

where $\alpha = 17280\sqrt{3026574721}$ and $\beta = 1920\sqrt{223572801841}$. We have checked that the first 19 Hecke eigenvalues of ψ_{21,D_3} and ϕ_{21,D_3} at odd positive integers match the Fourier coefficients of f_{38} and $\overline{f}_{38}$, respectively, and these newforms are elements of $M^-_{38}(2)$.

3.3.2.3. *The lattice D_5.* For the lattice D_5, Remarks 3.12 and 3.16 suggest that there exists a lifting map

$$J_{k,D_5} \to M^+_{2k-6}(2).$$

Note that $M^+_{2k-6}(2)$ is spanned by modular forms with eigenvalue -1 with respect to Atkin–Lehner involutions when k is even and by modular forms with eigenvalue $+1$ with respect to Atkin–Lehner involutions when k is odd.

Equation (3.19) implies that $J_{4,D_5} = \mathbb{C}E_{4,D_5}$ and this space should be mapped to $M_2(2)$. In particular, the Jacobi form E_{4,D_5} is a Hecke eigenform. We have checked that the first 41 Hecke eigenvalues of E_{4,D_5} at odd positive integers match the Fourier coefficients of $-E_2/24$. Note that $M_2(2) = \mathbb{C}(E_2(\tau) - 2E_2(2\tau))$.

Equation (3.19) implies that $J_{6,D_5} = \mathbb{C}E_{6,D_5}$ and this space should be mapped to $M_6(2)$. In particular, the Jacobi form E_{6,D_5} is a Hecke eigenform. We have checked that the first 41 Hecke eigenvalues of E_{6,D_5} at odd positive integers match the Fourier coefficients of $-E_6/504$.

Equation (3.18) implies that $J_{7,D_5} = \mathbb{C}\psi_{7,D_5}$ and this space should be mapped to $M_8(2)$. In particular, the Jacobi form ψ_{7,D_5} is a Hecke eigenform. The first few Fourier coefficients of ψ_{7,D_5} are listed in the fourth column of Table A.3. The space $M_8(2)$ contains precisely one newform:

$$\begin{aligned}
f_8(\tau) = &q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 - 512q^8 - 2043q^9\\
&+ 1680q^{10} + 1092q^{11} + 768q^{12} + 1382q^{13} - 8128q^{14} - 2520q^{15} + O(q^{16}).
\end{aligned}$$

The first 41 Hecke eigenvalues of ψ_{7,D_5} at odd positive integers match the Fourier coefficients of f_8 and this newform is an element of $M^+_8(2)$. In addition, $f_8(\tau) = \eta(\tau)^8\eta(2\tau)^8$ is an *eta product*.

Equation (3.19) implies that

$$J_{8,D_5} = \mathbb{C}E_4E_{4,D_5} \oplus \mathbb{C}E_{8,D_5}$$

and this space should be mapped to $M_{10}(2)$. Equation (3.22) implies that

$$\psi_{8,D_5}(\tau, z) := E_4(\tau)E_{4,D_5}(\tau, z) - \frac{192}{7}E_{8,D_5}(\tau, z)$$

is a Hecke eigenform in S_{8,D_5}. The first few Fourier coefficients of ψ_{8,D_5} are listed in Table A.4. The space $M_{10}(2)$ contains precisely one newform:

$$\begin{aligned} f_{10}(\tau) =& q + 16q^2 - 156q^3 + 256q^4 + 870q^5 - 2496q^6 - 952q^7 + 4096q^8 + 4653q^9 \\ &+ 13920q^{10} - 56148q^{11} - 39936q^{12} + 178094q^{13} - 15232q^{14} + O(q^{15}). \end{aligned}$$

The first 41 Hecke eigenvalues of ψ_{8,D_5} at odd positive integers match the Fourier coefficients of f_{10} and this newform is an element of $M^+_{10}(2)$.

Equation (3.18) implies that $J_{9,D_5} = \{0\}$ and this space should be mapped to $M_{12}(2)$. We have seen that the latter space contains no newforms.

Equation (3.19) implies that

$$J_{10,D_5} = \mathbb{C}E_6E_{4,D_5} \oplus \mathbb{C}E_4E_{6,D_5}$$

and this space should be mapped to $M_{14}(2)$. Equation (3.23) implies that

$$\psi_{10,D_5}(\tau, z) := \frac{3}{24}E_6(\tau)E_{4,D_5}(\tau, z) + E_4(\tau)E_{6,D_5}(\tau, z)$$

is a Hecke eigenform in S_{10,D_5}. The first few Fourier coefficients of ψ_{10,D_5} are listed in Table A.5. We have seen that the space $M_{14}(2)$ contains precisely two newforms. The first 41 Hecke eigenvalues of ψ_{10,D_5} at odd positive integers match the Fourier coefficients of g_{14} and this newform is an element of $M^+_{14}(2)$.

Equation (3.18) implies that $J_{11,D_5} = \mathbb{C}E_4\psi_{7,D_5}$ and this space should be mapped to $M_{16}(2)$. Set $\psi_{11,D_5} := E_4\psi_{7,D_5}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that the space $M_{16}(2)$ contains precisely one newform and we have checked that the first 41 Hecke eigenvalues of ψ_{11,D_5} at odd positive integers match the Fourier coefficients of f_{16}.

Equation (3.19) implies that

$$J_{12,D_5} = \mathbb{C}E_8E_{4,D_5} \oplus \mathbb{C}E_6E_{6,D_5} \oplus \mathbb{C}E_4E_{8,D_5}$$

and this space should be mapped to $M_{18}(2)$. Set

$$\begin{aligned} \beta^1_{12,5} :=& E_8E_{4,D_5} - \frac{192}{7}E_4E_{8,D_5} \text{ and} \\ \beta^2_{12,5} :=& E_6E_{6,D_5} + \frac{24}{7}E_4E_{8,D_5}. \end{aligned}$$

Equation (3.24) implies that these functions form a basis of S_{12,D_5}. Using Sage, we obtain that

$$T(3) = \begin{pmatrix} -\frac{15012}{7} & -\frac{30720}{7} \\ -\frac{28080}{7} & \frac{27612}{7} \end{pmatrix}$$

on this basis and this matrix can be diagonalized as

$$T(3) = \begin{pmatrix} 1 & 1 \\ -\frac{15}{8} & \frac{39}{80} \end{pmatrix} \begin{pmatrix} 6084 & 0 \\ 0 & -4284 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -\frac{15}{8} & \frac{39}{80} \end{pmatrix}^{-1}.$$

It follows that the two Jacobi forms ψ_{12,D_5} and ϕ_{12,D_5} defined by the system of equations

$$\begin{pmatrix} \psi_{12,D_5} \\ \phi_{12,D_5} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ -\frac{15}{8} & \frac{39}{80} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{12,5} \\ \beta^2_{12,5} \end{pmatrix} = \begin{pmatrix} \frac{13}{63}\beta^1_{12,5} - \frac{80}{189}\beta^2_{12,5} \\ \frac{50}{63}\beta^1_{12,5} + \frac{80}{189}\beta^2_{12,5} \end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{12,D_5}}(3) = 6084$ and $\lambda_{\phi_{12,D_5}}(3) = -4284$, respectively. We have seen that $M_{18}(2)$ contains precisely one newform and we have checked that the first 23 Hecke eigenvalues of ψ_{12,D_5} at odd positive integers match the Fourier coefficients of f_{18}. We have checked that the first 23 Hecke eigenvalues of ϕ_{12,D_5} at odd positive integers match the Fourier coefficients of the oldform F_{18}.

Equation (3.18) implies that $J_{13,D_5} = \mathbb{C}E_6\psi_{7,D_5}$ and this space should be mapped to $M_{20}(2)$. Set $\psi_{13,D_5} := E_6\psi_{7,D_5}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that the space $M_{20}(2)$ contains precisely two newforms. We have checked that the first 41 Hecke eigenvalues of ψ_{13,D_5} at odd positive integers match the Fourier coefficients of g_{20} and this newform is an element of $M^+_{20}(2)$.

Equation (3.19) implies that

$$J_{14,D_5} = \mathbb{C}E_{10}E_{4,D_5} \oplus \mathbb{C}E_8E_{6,D_5} \oplus \mathbb{C}E_6E_{8,D_5}$$

and this space should be mapped to $M_{22}(2)$. Set

$$\begin{aligned}\beta^1_{14,5} :=& E_{10}E_{4,D_5} - \frac{192}{7}E_6E_{8,D_5} \text{ and}\\ \beta^2_{14,5} :=& E_8E_{6,D_5} + \frac{24}{7}E_6E_{8,D_5}.\end{aligned}$$

Equation (3.25) implies that these functions form a basis of S_{14,D_5}. Following the same argument as in the weight 12 case, the two Jacobi forms ψ_{14,D_5} and ϕ_{14,D_5} defined by the system of equations

$$\begin{pmatrix}\psi_{14,D_5}\\ \phi_{14,D_5}\end{pmatrix} := \begin{pmatrix} 1 & 0\\ \frac{3}{8} & 1\end{pmatrix}^{-1}\begin{pmatrix}\beta^1_{14,5}\\ \beta^2_{14,5}\end{pmatrix}$$

are Hecke eigenforms of $T(3)$. We have seen that the space $M_{22}(2)$ contains precisely two newforms. We have checked that the first 23 Hecke eigenvalues of ψ_{14,D_5} at odd positive integers match the Fourier coefficients of g_{22} and this newform is an element of $M^+_{22}(2)$. The first 23 Hecke eigenvalues of ϕ_{14,D_1} at odd positive integers match the Fourier coefficients of the oldform F_{22}.

Equation (3.18) implies that $J_{15,D_5} = \mathbb{C}E_8\psi_{7,D_5}$ and this space should be mapped to $M_{24}(2)$. Set $\psi_{15,D_5} := E_8\psi_{7,D_5}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that the space $M_{24}(2)$ contains precisely one newform and we have checked that the first 41 Hecke eigenvalues of ψ_{15,D_5} at odd positive integers match the Fourier coefficients of f_{24}.

Equation (3.19) implies that

$$J_{16,D_5} = \mathbb{C}\Delta E_{4,D_5} \oplus \mathbb{C}E_{12}E_{4,D_5} \oplus \mathbb{C}E_{10}E_{6,D_5} \oplus \mathbb{C}E_8E_{8,D_5}$$

and this space should be mapped to $M_{26}(2)$. Set

$$\begin{aligned}\beta^1_{16,5} :=& E_{12}E_{4,D_5} - \frac{192}{7}E_8E_{8,D_5},\\ \beta^2_{16,5} :=& E_{10}E_{6,D_5} + \frac{24}{7}E_8E_{8,D_5} \text{ and}\\ \beta^3_{16,5} :=& \Delta E_{4,D_5}.\end{aligned}$$

Equation (3.26) implies that these functions form a basis of S_{16,D_5}. Using Sage, we obtain that

$$T(3) = \begin{pmatrix} -\frac{1202318412}{4837} & \frac{1669724160}{4837} & \frac{190577840947200}{477481}\\ \frac{6217200}{7} & -\frac{9330948}{7} & \frac{821653217280}{691}\\ 799 & -496 & \frac{1220032044}{691}\end{pmatrix}$$

on this basis and this matrix can be diagonalized as

$$\begin{aligned}T(3) = &\begin{pmatrix} 1 & 1 & 1 \\ \frac{-382123\sqrt{106705}-25405997}{59648240} & \frac{382123\sqrt{106705}-25405997}{59648240} & \frac{32477}{67280} \\ \frac{-3342367\sqrt{106705}+756027937}{515360793600} & \frac{3342367\sqrt{106705}+756027937}{515360793600} & -\frac{691}{2422080} \end{pmatrix} \\ &\times \begin{pmatrix} -4800\sqrt{106705}+189924 & 0 & 0 \\ 0 & 4800\sqrt{106705}+189924 & 0 \\ 0 & 0 & -195804 \end{pmatrix} \\ &\times \begin{pmatrix} 1 & 1 & 1 \\ \frac{-382123\sqrt{106705}-25405997}{59648240} & \frac{382123\sqrt{106705}-25405997}{59648240} & \frac{32477}{67280} \\ \frac{-3342367\sqrt{106705}+756027937}{515360793600} & \frac{3342367\sqrt{106705}+756027937}{515360793600} & -\frac{691}{2422080} \end{pmatrix}^{-1}.\end{aligned}$$

It follows that the three Jacobi forms ψ_{16,D_5}, ϕ_{16,D_5} and δ_{16,D_5} defined by the system of equations

$$\begin{pmatrix} \psi_{16,D_5} \\ \phi_{16,D_5} \\ \delta_{16,D_5} \end{pmatrix} := \begin{pmatrix} 1 & 1 & 1 \\ \frac{-382123\sqrt{106705}-25405997}{59648240} & \frac{382123\sqrt{106705}-25405997}{59648240} & \frac{32477}{67280} \\ \frac{-3342367\sqrt{106705}+756027937}{515360793600} & \frac{3342367\sqrt{106705}+756027937}{515360793600} & -\frac{691}{2422080} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{16,5} \\ \beta^2_{16,5} \\ \beta^3_{16,5} \end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{16,D_5}}(3) = -4800\sqrt{106705} + 189924$, $\lambda_{\phi_{16,D_5}}(3) = 4800\sqrt{106705} + 189924$ and $\lambda_{\delta_{16,D_5}}(3) = -195804$, respectively. We have seen that the space $M_{26}(2)$ contains precisely three newforms and we have checked that the first 23 Hecke eigenvalues of ψ_{16,D_5} and ϕ_{16,D_5} at odd positive integers match the Fourier coefficients of g_{26} and $\overline{g}_{26}$, respectively. We have checked that the first 23 Hecke eigenvalues of δ_{16,D_5} at odd positive integers match the Fourier coefficients of the oldform F_{26}.

Equation (3.18) implies that $J_{17,D_5} = \mathbb{C}E_{10}\psi_{7,D_5}$ and this space should be mapped to $M_{28}(2)$. Set $\psi_{17,D_5} := E_{10}\psi_{7,D_5}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that the space $M_{28}(2)$ contains precisely two newforms. We have checked that the first 41 Hecke eigenvalues of ψ_{17,D_5} at odd positive integers match the Fourier coefficients of g_{28} and this newform is an element of $M^+_{28}(2)$.

Equation (3.18) implies that

$$J_{19,D_5} = \mathbb{C}E_{12}\psi_{7,D_5} \oplus \mathbb{C}\Delta\psi_{7,D_5}$$

and this space should be mapped to $M_{32}(2)$. Set

$$\begin{aligned}\beta^1_{19,5} &:= E_{12}\psi_{7,D_5} \text{ and} \\ \beta^2_{19,5} &:= \Delta\psi_{7,D_5}.\end{aligned}$$

The matrix of $T(l)$ on J_{19,D_5} satisfies

$$\begin{pmatrix} T(l)\beta^1_{19,5} \\ T(l)\beta^2_{19,5} \end{pmatrix} = T(l)\begin{pmatrix} \beta^1_{19,5} \\ \beta^2_{19,5} \end{pmatrix}$$

and it can be computed following the same reasoning as before. Consequently, the two Jacobi forms ψ_{19,D_5} and ϕ_{19,D_5} defined by the system of equations

$$\begin{pmatrix} \psi_{19,D_5} \\ \phi_{19,D_5} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{477481\sqrt{987507049}-11643202817}{363831333465600} & \frac{-477481\sqrt{987507049}-11643202817}{363831333465600} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{19,5} \\ \beta^2_{19,5} \end{pmatrix}$$

are Hecke eigenforms. We have seen that $M_{32}(2)$ contains precisely three newforms. We have checked that the first 41 Hecke eigenvalues of ψ_{19,D_5} and ϕ_{19,D_5} at odd positive integers match the Fourier coefficients of g_{32} and $\overline{g}_{32}$, respectively, and these newforms are elements of $M^+_{32}(2)$.

3.3.2.4. *The lattice* D_7. For the lattice D_7, Remarks 3.12 and 3.16 suggest that there exists a lifting map

$$J_{k,D_7} \to M^+_{2k-8}(2).$$

Note that $M^+_{2k-8}(2)$ is spanned by modular forms with eigenvalue -1 with respect to Atkin–Lehner involutions when k is odd and by modular forms with eigenvalue $+1$ with respect to Atkin–Lehner involutions when k is even.

Equation (3.18) implies that $J_{5,D_7} = \mathbb{C}\psi_{5,D_7}$. In particular, the Jacobi form ψ_{5,D_7} is a Hecke eigenform. We have checked that the first 41 Hecke eigenvalues of ψ_{5,D_7} at odd positive integers match the Fourier coefficients of $-E_2/24$.

Equation (3.19) implies that $J_{6,D_7} = \mathbb{C}E_{6,D_7}$ and this space should be mapped to $M_4(2)$. In particular, the Jacobi form E_{6,D_7} is a Hecke eigenform. We have checked that the first 41 Hecke eigenvalues of E_{6,D_7} at odd positive integers match the Fourier coefficients of $E_4/240$.

Equation (3.18) implies that $J_{7,D_7} = \{0\}$ and this space should be mapped to $M_6(2)$. We have seen that the latter space contains no newforms.

Equation (3.19) implies that

$$J_{8,D_7} = \mathbb{C}E_4E_{4,D_7} \oplus \mathbb{C}E_{8,D_7}$$

and this space should be mapped to $M_8(2)$. Equation (3.22) implies that

$$\psi_{8,D_7}(\tau,z) := E_4(\tau)E_{4,D_7}(\tau,z) - \frac{576}{5}E_{8,D_7}(\tau,z)$$

is a Hecke eigenform in S_{8,D_7}. The first few Fourier coefficients of $5\psi_{8,D_7}$ are listed in Table A.6. We have seen that the space $M_8(2)$ contains precisely one newform. The first 41 Hecke eigenvalues of ψ_{8,D_7} at odd positive integers match the Fourier coefficients of f_8.

Equation (3.18) implies that $J_{9,D_7} = \mathbb{C}E_4\psi_{5,D_7}$ and this space should be mapped to $M_{10}(2)$. Set $\psi_{9,D_7} := E_4\psi_{5,D_7}$. In particular, this Jacobi form is a Hecke eigenform. The first few Fourier coefficients of ψ_{9,D_7} are listed in the second column of Table A.8. We have seen that the space $M_{10}(2)$ contains precisely one newform and the first few Hecke eigenvalues of ψ_{9,D_7} at odd positive integers match the Fourier coefficients of f_{10}.

Equation (3.19) implies that

$$J_{10,D_7} = \mathbb{C}E_6E_{4,D_7} \oplus \mathbb{C}E_4E_{6,D_7}$$

and this space should be mapped to $M_{12}(2)$. Equation (3.17) implies that

$$\psi_{10,D_7}(\tau,z) := \frac{1}{24}E_6(\tau)E_{4,D_7}(\tau,z) + E_4(\tau)E_{6,D_7}(\tau,z)$$

is a Hecke eigenform in S_{10,D_7}. The first few Fourier coefficients of $4\psi_{10,D_7}$ are listed in Table A.7. We have seen that the space $M_{12}(2)$ contains no newforms and the first 41 Hecke eigenvalues of ψ_{10,D_7} at odd positive integers match the Fourier coefficients of Δ.

Equation (3.18) implies that $J_{11,D_7} = \mathbb{C}E_6\psi_{5,D_7}$ and this space should be mapped to $M_{14}(2)$. Set $\psi_{11,D_7} := E_6\psi_{5,D_7}$. In particular, this Jacobi form is a Hecke eigenform. The first few Fourier coefficients of ψ_{11,D_7} are listed in the fourth column of Table A.8. We have seen that the space $M_{14}(2)$ contains precisely two newforms and the first 41 Hecke eigenvalues of ψ_{11,D_7} at odd positive integers match the Fourier coefficients of g_{14}.

Equation (3.19) implies that

$$J_{12,D_7} = \mathbb{C}E_8E_{4,D_7} \oplus \mathbb{C}E_6E_{6,D_7} \oplus \mathbb{C}E_4E_{8,D_7}$$

and this space should be mapped to $M_{16}(2)$. Set

$$\beta^1_{12,7} := E_8 E_{4,D_7} - \frac{576}{5} E_4 E_{8,D_7} \text{ and}$$
$$\beta^2_{12,7} := E_6 E_{6,D_7} + \frac{24}{5} E_4 E_{8,D_7}.$$

Equation (3.24) implies that these functions form a basis of S_{12,D_7}. Using Sage, we obtain that

$$T(3) = \begin{pmatrix} 25452 & -829440 \\ 2000/3 & -22548 \end{pmatrix}$$

on this basis and this matrix can be diagonalized as

$$T(3) = \begin{pmatrix} 1 & 1 \\ \frac{5}{216} & \frac{5}{144} \end{pmatrix} \begin{pmatrix} 6252 & 0 \\ 0 & -3348 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{5}{216} & \frac{5}{144} \end{pmatrix}^{-1}.$$

It follows that the two Jacobi forms ψ_{12,D_7} and ϕ_{12,D_7} defined by the system of equations

$$\begin{pmatrix} \psi_{12,D_7} \\ \phi_{12,D_7} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{5}{216} & \frac{5}{144} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{12,7} \\ \beta^2_{12,7} \end{pmatrix} = \begin{pmatrix} 3\beta^1_{12,7} - \frac{432}{5}\beta^2_{12,7} \\ -2\beta^1_{12,7} + \frac{432}{5}\beta^2_{12,7} \end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{12,D_7}}(3) = 6252$ and $\lambda_{\phi_{12,D_7}}(3) = -3348$, respectively. We have seen that $M_{16}(2)$ contains precisely one newform and we have checked that the first 23 Hecke eigenvalues of ψ_{12,D_7} at odd positive integers match the Fourier coefficients of f_{16}. We have checked that the first 23 Hecke eigenvalues of ϕ_{12,D_7} at odd positive integers match the Fourier coefficients of the oldform F_{16}.

Equation (3.18) implies that $J_{13,D_7} = \mathbb{C}E_8\psi_{5,D_7}$ and this space should be mapped to $M_{18}(2)$. Set $\psi_{13,D_7} := E_8\psi_{5,D_7}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that the space $M_{18}(2)$ contains precisely one newform and we have checked that the first 41 Hecke eigenvalues of ψ_{13,D_7} at odd positive integers match the Fourier coefficients of f_{18}.

Equation (3.19) implies that

$$J_{14,D_7} = \mathbb{C}E_{10}E_{4,D_7} \oplus \mathbb{C}E_8 E_{6,D_7} \oplus \mathbb{C}E_6 E_{8,D_7}$$

and this space should be mapped to $M_{20}(2)$. Set

$$\beta^1_{14,7} := E_{10} E_{4,D_7} - \frac{576}{5} E_6 E_{8,D_7} \text{ and}$$
$$\beta^2_{14,7} := E_8 E_{6,D_7} + \frac{24}{5} E_6 E_{8,D_7}.$$

Equation (3.25) implies that these functions form a basis of S_{14,D_7}. Following the same argument as in the weight 12 case, the two Jacobi forms ψ_{14,D_7} and ϕ_{14,D_7} defined by the system of equations

$$\begin{pmatrix} \psi_{14,D_7} \\ \phi_{14,D_7} \end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{89}{3024} & \frac{77}{2952} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{14,7} \\ \beta^2_{14,7} \end{pmatrix}$$

are Hecke eigenforms of $T(3)$. We have seen that the space $M_{20}(2)$ contains precisely two newforms and we have checked that the first 23 Hecke eigenvalues of ϕ_{14,D_7} at odd positive integers match the Fourier coefficients of g_{20}. The first 23 Hecke eigenvalues of ψ_{14,D_7} at odd positive integers match the Fourier coefficients of the oldform F_{20}.

Equation (3.18) implies that $J_{15,D_7} = \mathbb{C}E_{10}\psi_{5,D_7}$ and this space should be mapped to $M_{22}(2)$. Set $\psi_{15,D_7} := E_{10}\psi_{5,D_7}$. In particular, this Jacobi form is a Hecke eigenform. We have seen that the space $M_{22}(2)$ contains precisely two newforms. We have checked that the first 41 Hecke eigenvalues of ψ_{15,D_7} at odd positive integers match the Fourier coefficients of g_{22} and this newform is an element of $M^+_{22}(2)$.

Equation (3.19) implies that

$$J_{16,D_7} = \mathbb{C}\Delta E_{4,D_7} \oplus \mathbb{C}E_{12}E_{4,D_7} \oplus \mathbb{C}E_{10}E_{6,D_7} \oplus \mathbb{C}E_8E_{8,D_7}$$

and this space should be mapped to $M_{24}(2)$. Set

$$\begin{aligned}
\beta^1_{16,7} :=& E_{12}E_{4,D_7} - \frac{576}{5}E_8E_{8,D_7},\\
\beta^2_{16,7} :=& E_{10}E_{6,D_7} + \frac{24}{5}E_8E_{8,D_7} \text{ and}\\
\beta^3_{16,7} :=& \Delta E_{4,D_7}.
\end{aligned}$$

Equation (3.26) implies that these functions form a basis of S_{16,D_7}. Using Sage, we obtain that

$$T(3) = \begin{pmatrix} \frac{3889276452}{691} & -\frac{135604316160}{691} & \frac{79595230118215680}{477481} \\ 158160 & -5517108 & \frac{3162724392960}{691} \\ \frac{83}{3} & -976 & -\frac{191956572}{691} \end{pmatrix}$$

on this basis and this matrix can be diagonalized as

$$\begin{aligned}
T(3) =& \begin{pmatrix} 1 & 1 & 1 \\ \frac{21421\sqrt{144169}}{18929980896} + \frac{176954735}{6309993632} & -\frac{21421\sqrt{144169}}{18929980896} + \frac{176954735}{6309993632} & \frac{3455}{126828} \\ \frac{-477481\sqrt{144169}+60215813}{224888173044480} & \frac{477481\sqrt{144169}+60215813}{224888173044480} & -\frac{691}{146105856} \end{pmatrix}\\
&\times \begin{pmatrix} -576\sqrt{144169}+169740 & 0 & 0 \\ 0 & 576\sqrt{144169}+169740 & 0 \\ 0 & 0 & -505908 \end{pmatrix}\\
&\times \begin{pmatrix} 1 & 1 & 1 \\ \frac{21421\sqrt{144169}}{18929980896} + \frac{176954735}{6309993632} & -\frac{21421\sqrt{144169}}{18929980896} + \frac{176954735}{6309993632} & \frac{3455}{126828} \\ \frac{-477481\sqrt{144169}+60215813}{224888173044480} & \frac{477481\sqrt{144169}+60215813}{224888173044480} & -\frac{691}{146105856} \end{pmatrix}^{-1}.
\end{aligned}$$

It follows that the three Jacobi forms ψ_{16,D_7}, ϕ_{16,D_7} and δ_{16,D_7} defined by the system of equations

$$\begin{pmatrix} \psi_{16,D_7} \\ \phi_{16,D_7} \\ \delta_{16,D_7} \end{pmatrix} := \begin{pmatrix} 1 & 1 & 1 \\ \frac{21421\sqrt{144169}}{18929980896} + \frac{176954735}{6309993632} & -\frac{21421\sqrt{144169}}{18929980896} + \frac{176954735}{6309993632} & \frac{3455}{126828} \\ \frac{-477481\sqrt{144169}+60215813}{224888173044480} & \frac{477481\sqrt{144169}+60215813}{224888173044480} & -\frac{691}{146105856} \end{pmatrix}^{-1} \begin{pmatrix} \beta^1_{16,7} \\ \beta^2_{16,7} \\ \beta^3_{16,7} \end{pmatrix}$$

are Hecke eigenforms of $T(3)$, with eigenvalues $\lambda_{\psi_{16,D_7}}(3) = -576\sqrt{144169} + 169740$, $\lambda_{\phi_{16,D_7}}(3) = 576\sqrt{144169} + 169740$ and $\lambda_{\delta_{16,D_7}}(3) = -505908$, respectively. We have seen that the space $M_{24}(2)$ contains precisely one newform and we have checked that the first 23 Hecke eigenvalues of δ_{16,D_7} at odd positive integers match the Fourier coefficients of f_{24}. We have checked that the first 23 Hecke eigenvalues of ψ_{16,D_7} and ϕ_{16,D_7} at odd positive integers match the Fourier coefficients of the oldforms F_{24} and $\overline{F}_{24}$, respectively.

Equation (3.18) implies that

$$J_{17,D_7} = \mathbb{C}E_{12}\psi_{5,D_7} \oplus \mathbb{C}\Delta\psi_{5,D_7}$$

and this space should be mapped to $M_{26}(2)$. Set

$$\beta^1_{17,7} := E_{12}\psi_{5,D_7} \qquad \text{and} \qquad \beta^2_{17,7} := \Delta\psi_{5,D_7}.$$

The matrix of $T(l)$ on J_{17,D_7} satisfies

$$\begin{pmatrix} T(l)\beta^1_{17,7} \\ T(l)\beta^2_{17,7} \end{pmatrix} = T(l)\begin{pmatrix} \beta^1_{17,7} \\ \beta^2_{17,7} \end{pmatrix}$$

and it can be computed following the same reasoning as before. Consequently, the two Jacobi forms ψ_{17,D_7} and ϕ_{17,D_7} defined by the system of equations

$$\begin{pmatrix}\psi_{17,D_7}\\ \phi_{17,D_7}\end{pmatrix} := \begin{pmatrix} 1 & 1 \\ \frac{-477481\sqrt{106705}+33963341}{232962188659200} & \frac{477481\sqrt{106705}+33963341}{232962188659200}\end{pmatrix}^{-1}\begin{pmatrix}\beta^1_{17,7}\\ \beta^2_{17,7}\end{pmatrix}$$

are Hecke eigenforms. We have seen that $M_{26}(2)$ contains precisely three newforms. We have checked that the first 41 Hecke eigenvalues of ϕ_{17,D_7} and ψ_{17,D_7} at odd positive integers match the Fourier coefficients of g_{26} and $\overline{g}_{26}$, respectively, and these newforms are elements of $M^+_{26}(2)$.

3.3.3. Concluding remarks. The results of this section are summarized in Table 3.1. Modular forms whose first few Hecke eigenvalues match are listed on the same row. Elliptic Eisenstein series together with the corresponding Jacobi forms are marked in green, elliptic newforms in the + space together with the corresponding Jacobi forms are marked in blue, elliptic newforms in the − space together with the corresponding Jacobi forms are marked in orange and elliptic oldforms together with their corresponding Jacobi forms are marked in red.

Theorem 3.29 contradicts Conjecture 6.1.3 in [**Ajo15**], which would imply that

$$J_{k+1,D_1} \simeq J_{k+2,D_3} \simeq \mathfrak{M}^-_{2k}(2) \text{ and}$$
$$J_{k+3,D_5} \simeq J_{k+4,D_7} \simeq \mathfrak{M}^+_{2k}(2),$$

since, for example, $J_{12,D_1} \not\simeq J_{13,D_3}$. However, the conjectured weight and level seem to be correct. Furthermore, Jacobi forms of index D_1 and D_3 which correspond to newforms indeed map to the − space and Jacobi forms of index D_5 and D_7 which correspond to newforms map to the + space. Let J^{old}_{k,D_n} and J^{new}_{k,D_n} denote the subspaces of Jacobi forms in J_{k,D_n} whose Hecke eigenvalues match the Hecke eigenvalues of elliptic oldforms and elliptic newfroms, respectively. In particular, the results in Table 3.1 suggest that

$$J^{old}_{k+1,D_1} \simeq J^{old}_{k+3,D_5} \simeq \begin{cases} 0, & \text{if } k \text{ is even and} \\ M_{2k}(1), & \text{if } k \text{ is odd,}\end{cases}$$
$$J^{old}_{k+2,D_3} \simeq J^{old}_{k+4,D_7} \simeq \begin{cases} M_{2k}(1), & \text{if } k \text{ is even and} \\ 0, & \text{if } k \text{ is odd,}\end{cases}$$
$$J^{new}_{k+1,D_1} \simeq J^{new}_{k+2,D_3} \simeq M^{-,new}_{2k}(2),$$
$$J^{new}_{k+3,D_5} \simeq J^{new}_{k+4,D_7} \simeq M^{+,new}_{2k}(2).$$

Theorem 1.37 implies that

$$J_{k+1,D_1} \simeq M^{new,-}_{2k}(2) \oplus M^-_{2k}(1)$$

as Hecke modules. The remaining above statements can be re-formulated as the following:

Conjecture 3.30. *For every $k \geq 2$, the following holds:*

$$J_{k+2,D_3} \simeq M^{new,-}_{2k}(2) \oplus M^+_{2k}(1),$$
$$J_{k+3,D_5} \simeq M^{new,+}_{2k}(2) \oplus M^-_{2k}(1),$$
$$J_{k+4,D_7} \simeq M^{new,+}_{2k}(2) \oplus M^+_{2k}(1)$$

and these isomorphisms are Hecke equivariant.

TABLE 3.1. Correspondence between Jacobi forms of index D_n ($n = 1, 3, 5$ and 7) and elliptic modular forms

k	J_{k+1,D_1}	J_{k+2,D_3}	$M_{2k}(2)$	J_{k+3,D_5}	J_{k+4,D_7}
2		E_{4,D_3}	E_4		E_{6,D_7}
3	E_{4,D_1}		E_6	E_{6,D_5}	
4		E_{6,D_3}	E_8 f_8	ψ_{7,D_5}	ψ_{8,D_7}
5	E_{6,D_1}		E_{10} f_{10}	ψ_{8,D_5}	ψ_{9,D_7}
6		ψ_{8,D_3}	Δ		ψ_{10,D_7}
7	ψ_{8,D_1}	ψ_{9,D_3}	f_{14} g_{14}	ψ_{10,D_5}	ψ_{11,D_7}
8		ψ_{10,D_3}	f_{16} F_{16}	ψ_{11,D_5}	ψ_{12,D_7} ϕ_{12,D_7}
9	ψ_{10,D_1}		f_{18} F_{18}	ψ_{12,D_5} ϕ_{12,D_5}	ψ_{13,D_7}
10	ψ_{11,D_1}	ϕ_{12,D_3} ψ_{12,D_3}	f_{20} g_{20} F_{20}	ψ_{13,D_5}	ϕ_{14,D_7} ψ_{14,D_7}
11	ψ_{12,D_1} ϕ_{12,D_1}	ψ_{13,D_3}	f_{22} g_{22} F_{22}	ψ_{14,D_5} ϕ_{14,D_5}	ψ_{15,D_7}
12		ψ_{14,D_3} ϕ_{14,D_3}	f_{24} F_{24} $\overline{F}_{24}$	ψ_{15,D_5}	δ_{16,D_7} ψ_{16,D_7} ϕ_{16,D_7}
13	ψ_{14,D_1} ϕ_{14,D_1}	ψ_{15,D_3}	f_{26} g_{26} $\overline{g}_{26}$ F_{26}	ψ_{16,D_5} ϕ_{16,D_5} δ_{16,D_5}	ϕ_{17,D_7} ψ_{17,D_7}
14	ψ_{15,D_1}	δ_{16,D_3} ψ_{16,D_3} ϕ_{16,D_3}	f_{28} g_{28} F_{28} $\overline{F}_{28}$	ψ_{17,D_5}	
15	δ_{16,D_1} ϕ_{16,D_1} ψ_{16,D_1}	ψ_{17,D_3}	f_{30} g_{30} F_{30} $\overline{F}_{30}$		
16	ψ_{17,D_1}		f_{32} g_{32} $\overline{g}_{32}$	ψ_{19,D_5} ϕ_{19,D_5}	
17		ψ_{19,D_3}	f_{34}		
18	ψ_{19,D_1}		f_{36}		
19		ψ_{21,D_3} ϕ_{21,D_3}	f_{38} $\overline{f}_{38}$		
20	ψ_{21,D_1}		f_{40}		
22	ψ_{23,D_1} ϕ_{23,D_1}		f_{44} $\overline{f}_{44}$		

CHAPTER 4

Level raising operators

We define a generalization of the operators U_l and V_l from [**EZ85**, §I.4] for Jacobi forms of lattice index and study some of their properties. Given the terminology on one hand and the connection between Jacobi forms and elliptic modular forms conjectured in [**Ajo15**, §6.1.1] on the other, the *level* of a Jacobi form should be the level of the lattice in its index. This is supported by results from [**Sak18**], which state that the space of Jacobi newforms of weight k and scalar index 1 for $\Gamma_0(N)$ which is invariant with respect to the action of a certain Atkin–Lehner operator is isomorphic to the space of Jacobi newforms of weight k and scalar index N for Γ as modules over the Hecke algebra (for every odd, square-free N).

4.1. The U operators

These operators arise from *isometries* of lattices (see end of Section 1.2). We remind the reader that an isometry of a lattice into another is an injective linear map on the underlying quadratic modules, which preserves the bilinear forms.

Definition 4.1. Let $\underline{L}_1$ and $\underline{L}_2$ be positive-definite, even lattices over $\mathbb{Z}$ such that there exists and isometry σ of $\underline{L}_1$ into $\underline{L}_2$. Define a linear operator

$$U(\sigma) : J_{k,\underline{L}_2} \to \{\phi : \mathfrak{H} \times (L_1 \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C} : \phi \text{ is holomorphic}\}$$

as

$$U(\sigma)\phi(\tau, z_1) := \phi(\tau, \sigma(z_1)).$$

This operator satisfies the following:

Lemma 4.2. *Let σ be an isometry of $\underline{L}_1$ into $\underline{L}_2$. For every ϕ in $J_{k,\underline{L}_2}$, the function $U(\sigma)\phi$ is invariant with respect to the $|_{k,\underline{L}_1}$-action of $J^{\underline{L}_1}$.*

Proof. For every A in Γ, we have

$$\begin{aligned}(U(\sigma)\phi)|_{k,\underline{L}_1} A(\tau, z_1) =& U(\sigma)\phi\left(A\tau, \frac{z_1}{c\tau+d}\right)(c\tau+d)^{-k} e\left(\frac{-c\beta_1(z_1)}{c\tau+d}\right)\\ =&\phi\left(A\tau, \frac{\sigma(z_1)}{c\tau+d}\right)(c\tau+d)^{-k} e\left(\frac{-c\beta_2\left(\sigma(z_1)\right)}{c\tau+d}\right)\\ =&\phi|_{k,\underline{L}_2} A(\tau, \sigma(z_1)) = \phi(\tau, \sigma(z_1)) = U(\sigma)\phi(\tau, z_1),\end{aligned}$$

since $\beta_2 \circ \sigma = \beta_1$ and ϕ is a Jacobi form of weight k and index $\underline{L}_2$.

On the other hand, for every (λ, μ) in $H^{\underline{L}_1}(\mathbb{Z})$, we have

$$\begin{aligned}(U(\sigma)\phi)|_{\underline{L}_1}(\lambda, \mu)(\tau, z_1) =& U(\sigma)\phi(\tau, z_1 + \lambda\tau + \mu) e(\tau\beta_1(\lambda) + \beta_1(\lambda, z_1))\\ =&\phi\left(\tau, \sigma(z_1) + \tau\sigma(\lambda) + \sigma(\mu)\right) e\left(\tau\beta_2(\sigma(\lambda)) + \beta_2\left(\sigma(\lambda), \sigma(z_1)\right)\right)\\ =&\phi|_{\underline{L}_2}(\sigma(\lambda), \sigma(\mu))(\tau, \sigma(z_1))\\ =&\phi(\tau, \sigma(z_1)) = U(\sigma)\phi(\tau, z_1),\end{aligned}$$

since $\beta_2 \circ \sigma = \beta_1$ and ϕ is a Jacobi form of index $\underline{L}_2$. It follows that $U(\sigma)\phi$ is invariant under the $|_{k,\underline{L}_1}$-action of $J^{\underline{L}_1}$, as claimed. □

We would like for $U(\sigma)\phi$ to be a Jacobi form of weight k and index $\underline{L}_1$. If ϕ in $J_{k,\underline{L}_2}$ has a Fourier expansion of the type

$$\phi(\tau, z_2) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, z_2)),$$

then

$$U(\sigma)\phi(\tau, z_1) = \phi(\tau, \sigma(z_1)) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e\left(n\tau + \beta_2(r_2, \sigma(z_1))\right).$$

We need $\beta_2(r_2, \sigma(z_1)) = \beta_1(r_1, z_1)$ for some r_1 in $L_1^\#$ for every r_2 in $L_2^\#$ such that $c_\phi(n, r_2)$ is non-zero in order for $U(\sigma)\phi$ to have the correct Fourier expansion. One case in which this condition holds is when σ is surjective and we can make the change of variable $r' = \sigma^{-1}(r)$ in the above equation.

Assume that σ is surjective on $L_2^\#$. Then $\sigma : \sigma^{-1}(L_2^\#) \to L_2^\#$ is a $\mathbb{Z}$-module isomorphism and, furthermore,

$$\begin{aligned}
&\sigma^{-1}(L_2^\#) = \{r \in L_1 \otimes \mathbb{Q} : \beta_2(x, \sigma(r)) \in \mathbb{Z} \text{ for all } x \text{ in } L_2\} \\
&\implies \text{ for every } r \text{ in } \sigma^{-1}(L_2^\#), \beta_2(x, \sigma(r)) \in \mathbb{Z} \text{ for all } x \text{ in } \sigma(L_1) \\
&\iff \text{ for every } r \text{ in } \sigma^{-1}(L_2^\#), \beta_1(\sigma^{-1}(x), r) \in \mathbb{Z} \text{ for all } x \text{ in } \sigma(L_1) \\
&\iff \text{ for every } r \text{ in } \sigma^{-1}(L_2^\#), \beta_1(x', r) \in \mathbb{Z} \text{ for all } x' \text{ in } L_1 \\
&\implies \sigma^{-1}(L_2^\#) \subseteq L_1^\#.
\end{aligned}$$

This implies that $\sigma^{-1}(L_2^\#)$ is a $\mathbb{Z}$-submodule of $L_1^\#$ and hence that $\operatorname{rk}(\underline{L}_2) \leq \operatorname{rk}(\underline{L}_1)$. On the other hand, since σ is injective by definition, we also have that $\operatorname{rk}(\underline{L}_1) \leq \operatorname{rk}(\underline{L}_2)$. It follows that $\operatorname{rk}(\underline{L}_1) = \operatorname{rk}(\underline{L}_2)$, which is equivalent to the fact that $\sigma : L_1 \otimes \mathbb{Q} \to L_2 \otimes \mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-modules. Conversely, suppose that $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ satisfy $\operatorname{rk}(\underline{L}_1) = \operatorname{rk}(\underline{L}_2)$. Then every isometry σ of $\underline{L}_1$ into $\underline{L}_2$ is necessarily surjective as a map between $L_1 \otimes \mathbb{Q}$ and $L_2 \otimes \mathbb{Q}$. It follows that $\sigma : L_1 \otimes \mathbb{Q} \to L_2 \otimes \mathbb{Q}$ is an isomorphism of $\mathbb{Q}$-modules and therefore it is invertible on $L_2^\#$. Hence, every isometry σ of $\underline{L}_1$ into $\underline{L}_2$ is invertible on $L_2^\#$ if and only if $\operatorname{rk}(\underline{L}_1) = \operatorname{rk}(\underline{L}_2)$ (if and only if $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$). As a consequence, the following holds:

THEOREM 4.3. *Let $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ be positive-definite, even lattices over $\mathbb{Z}$, of the same rank and such that there exists an isometry σ of $\underline{L}_1$ into $\underline{L}_2$. Then $U(\sigma)$ maps $J_{k,\underline{L}_2}$ to $J_{k,\underline{L}_1}$. Furthermore, if ϕ in $J_{k,\underline{L}_2}$ has a Fourier expansion of the type*

$$\phi(\tau, z_2) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, z_2)),$$

then $U(\sigma)\phi$ has the following Fourier expansion:

$$U(\sigma)\phi(\tau, z_1) = \sum_{\substack{n\in\mathbb{Z}, r_1\in L_1^\# \\ n\geq\beta_1(r_1), \sigma(r_1)\in L_2^\#}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)).$$

Proof. Lemma 4.2 implies that $U(\sigma)\phi$ transforms like a Jacobi form of weight k and index $\underline{L}_1$. In light of the discussion above regarding Fourier expansions, we have

$$\begin{aligned} U(\sigma)\phi(\tau, z_1) &= \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n\geq\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, \sigma(z_1))) \\ &= \sum_{\substack{n\in\mathbb{Z}, r_1\in\sigma^{-1}(L_2^\#) \\ n\geq\beta_1(r_1)}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)) \\ &= \sum_{\substack{n\in\mathbb{Z}, r_1\in L_1^\# \\ n\geq\beta_1(r_1), \sigma(r_1)\in L_2^\#}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)), \end{aligned}$$

as claimed. □

Corollary 4.4. *Let $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ be positive-definite, even lattices over $\mathbb{Z}$, of the same rank and such that there exists an isometry σ of $\underline{L}_1$ into $\underline{L}_2$. Then $U(\sigma)$ maps $S_{k,\underline{L}_2}$ to $S_{k,\underline{L}_1}$.*

Proof. If ϕ in $S_{k,\underline{L}_2}$ and has a Fourier expansion of the type

$$\phi(\tau, z_2) = \sum_{\substack{n\in\mathbb{Z}, r_2\in L_2^\# \\ n>\beta_2(r_2)}} c_\phi(n, r_2)e(n\tau + \beta_2(r_2, z_2)),$$

then the above theorem implies that $U(\sigma)\phi$ has the following Fourier expansion:

$$U(\sigma)\phi(\tau, z_1) = \sum_{\substack{n\in\mathbb{Z}, r_1\in L_1^\# \\ \sigma(r_1)\in L_2^\#, n\geq\beta_1(r_1)}} c_\phi(n, \sigma(r_1))e(n\tau + \beta_1(r_1, z_1)).$$

If $n = \beta_1(r_1)$ in the above equation, then $n = \beta_2(\sigma(r_1))$ and hence $c_\phi(n, \sigma(r_1)) = 0$, since ϕ is a cusp form. It follows that $U(\sigma)\phi$ is also a cusp form. □

We will show that the $U(\cdot)$ operators preserve Eisenstein series in the following sections.

Remark 4.5. In Section 3.3, we encountered an example of an isometry of D_n ($n = 1, 3, 5$ and 7) into E_8 which is not surjective, but preserves Jacobi forms nonetheless:

$$\alpha_n : D_n \to E_8 : (x_1, \ldots, x_n) \mapsto (0, \ldots, 0, x_1, \ldots, x_n).$$

This is due to the fact that, for every ϕ in J_{k,E_8}, we have

$$U(\alpha_n)\phi(\tau, z) = \alpha_n^*\phi(\tau, z) = \phi(\tau, \alpha_n(z)) = \sum_{\substack{n\in\mathbb{Z}, r\in E_8, \\ n\geq\frac{(r,r)}{2}}} c_\phi(n, r)e\left(n\tau + (r, \alpha_n(z))\right)$$

and, for every r in E_8 and every z in $D_n \otimes \mathbb{C}$, there exists an r' in $D_n^\#$ such that

$$(r, \alpha_n(z)) = (r', z).$$

More precisely, we have

$$(r, \alpha_n(z)) = \left(\alpha_n\left((r_{8-n+1}, \ldots, r_8)\right), \alpha_n(z)\right) = ((r_{8-n+1}, \ldots, r_8), z)$$

and $(r_{8-n+1},\dots,r_8)\in D_n^\#$ for every $r=(r_1,\dots,r_8)$ in E_8 (see Example 1.6). It follows that

$$\begin{aligned} U(\alpha_n)\phi(\tau,z) &= \sum_{(r_{8-n+1},\dots,r_8)\in D_n^\#}\ \sum_{\substack{(r_1,\dots,r_{8-n})\\ (r_1,\dots,r_8)\in E_8}}\ \sum_{\substack{n\in\mathbb{Z}\\ n\ge \frac{r_1^2+\dots+r_8^2}{2}}} c_\phi(n,r)e\left(n\tau+((r_{8-n+1},\dots,r_8),z)\right)\\ &= \sum_{\substack{(r_{8-n+1},\dots,r_8)\in D_n^\#, n\in\mathbb{Z}\\ n\ge\frac{r_{8-n+1}^2+\dots+r_8^2}{2}}}\ \sum_{\substack{(r_1,\dots,r_{8-n})\\ (r_1,\dots,r_8)\in E_8, n\ge\frac{r_1^2+\dots+r_8^2}{2}}} c_\phi(n,r)e\left(n\tau+((r_{8-n+1},\dots,r_8),z)\right)\\ &= \sum_{\substack{n\in\mathbb{Z}, s\in D_n^\#\\ n\ge\frac{(s,s)}{2}}} c_{U(\alpha_n)\phi}(n,s)e(n\tau+(s,z)), \end{aligned}$$

where

$$c_{U(\alpha_n)\phi}(n,s)=\sum_{\substack{(x_1,\dots,x_{8-n})\\ (s_1,\dots,s_n,x_1,\dots,x_{8-n})\in E_8\\ n-\frac{s_1^2+\dots+s_n^2}{2}\ge\frac{x_1^2+\dots+x_{8-n}^2}{2}}} c_\phi\left(n,(s_1,\dots,s_n,x_1,\dots,x_{8-n})\right).$$

The operators $U(\cdot)$ *raise the level* of the index of the Jacobi form that they are applied to:

Lemma 4.6. *If $\underline{L}_1=(L_1,\beta_1)$ and $\underline{L}_2=(L_2,\beta_2)$ are positive-definite, even lattices over $\mathbb{Z}$, of the same rank and such that σ is an isometry of $\underline{L}_1$ into $\underline{L}_2$, then* $\mathrm{lev}(\underline{L}_2)\mid\mathrm{lev}(\underline{L}_1)$.

Proof. By definition,

$$\begin{aligned}\mathrm{lev}(\underline{L}_2)&=\min\{N\in\mathbb{N}:N\beta_2(r)\in\mathbb{Z}\text{ for all }r\text{ in }L_2^\#\}\\ &=\min\{N\in\mathbb{N}:N\beta_1(\sigma^{-1}(r))\in\mathbb{Z}\text{ for all }r\text{ in }L_2^\#\}.\end{aligned}$$

On the other hand, $\mathrm{lev}(\underline{L}_1)\beta_1(\sigma^{-1}(r))\in\mathbb{Z}$ for all r in $L_2^\#$. Hence, $\mathrm{lev}(\underline{L}_2)\mid\mathrm{lev}(\underline{L}_1)$. □

Example 4.7. The operator U_l defined in [**EZ85**] arises from the following isometry of the lattice $(\mathbb{Z},(x,y)\mapsto ml^2xy)$ into the lattice $(\mathbb{Z},(x,y)\mapsto mxy)$:

$$\sigma_l:(\mathbb{Q},(x,y)\mapsto ml^2xy)\to(\mathbb{Q},(x,y)\mapsto mxy),\quad \sigma_l(x)=lx.$$

It raises the level by a factor of l^2.

Fix any two bases for $L_1\otimes\mathbb{Q}$ and $L_2\otimes\mathbb{Q}$, let G_1 and G_2 denote the Gram matrices of $\underline{L}_1$ and $\underline{L}_2$, respectively, and let M denote the matrix of σ with respect to these bases. Then

$$\beta_2\circ\sigma=\beta_1\iff M^tG_2M=G_1\implies\det(\underline{L}_1)=\det(\underline{L}_2)\det(M)^2.$$

In other words, we have shown the following:

Lemma 4.8. *If $\underline{L}_1=(L_1,\beta_1)$ and $\underline{L}_2=(L_2,\beta_2)$ are positive-definite, even lattices over $\mathbb{Z}$ and σ is an isometry of $\underline{L}_1$ into $\underline{L}_2$, then* $\det(\underline{L}_1)=\det(\sigma)^2\det(\underline{L}_2)$.

We remind the reader that $\mathrm{lev}(\underline{L})$ and $\det(\underline{L})$ have the same set of prime divisors for every fixed positive-definite, even lattice $\underline{L}$. It follows from this fact and from Lemmas 4.6 and 4.8 that, when $L_1\otimes\mathbb{Q}\simeq L_2\otimes\mathbb{Q}$, the set of prime divisors of $\frac{\mathrm{lev}(\underline{L}_1)}{\mathrm{lev}(\underline{L}_2)}$ consists of the prime divisors of $\det(\sigma)$ which are not divisors of $\mathrm{lev}(\underline{L}_2)$, plus possibly some primes dividing $\mathrm{lev}(\underline{L}_2)$. Write

$$\mathrm{lev}(\underline{L}_1)\mid\delta\det(\underline{L}_1)\mid\mathrm{lev}(\underline{L}_1)^{\mathrm{rk}(\underline{L}_1)},$$

where

$$\delta := \begin{cases} 2, & \text{if } \operatorname{rk}(\underline{L}_1) \equiv 1 \bmod 2 \text{ and} \\ 1, & \text{otherwise.} \end{cases}$$

Then, writing $\operatorname{lev}(\underline{L}_1) = \operatorname{lev}(\underline{L}_2)\frac{\operatorname{lev}(\underline{L}_1)}{\operatorname{lev}(\underline{L}_2)}$ in the above, we obtain that

$$\lceil \frac{2v_p(\det(\sigma))}{\operatorname{rk}(\underline{L}_2)} \rceil \leq v_p\left(\frac{\operatorname{lev}(\underline{L}_1)}{\operatorname{lev}(\underline{L}_2)}\right) \leq 2v_p(\det(\sigma))$$

for primes dividing $\det(\sigma)$ which do not divide $\operatorname{lev}(\underline{L}_2)$. For primes dividing $\operatorname{lev}(\underline{L}_2)$, we obtain the bounds

$$\begin{aligned} &\lceil \frac{1 + v_p(\det(\underline{L}_2)) + 2v_p(\det(\sigma))}{\operatorname{rk}(\underline{L}_2)} \rceil - v_p(\operatorname{lev}(\underline{L}_2)) \leq v_p\left(\frac{\operatorname{lev}(\underline{L}_1)}{\operatorname{lev}(\underline{L}_2)}\right) \\ &\leq 1 + v_p(\det(\underline{L}_2)) - v_p(\operatorname{lev}(\underline{L}_2)) + 2v_p(\det(\sigma)) \end{aligned}$$

if $p = 2$ and $\operatorname{rk}(\underline{L}_2) \equiv 1 \bmod 2$ and

$$\begin{aligned} &\lceil \frac{v_p(\det(\underline{L}_2)) + 2v_p(\det(\sigma))}{\operatorname{rk}(\underline{L}_2)} \rceil - v_p(\operatorname{lev}(\underline{L}_2)) \leq v_p\left(\frac{\operatorname{lev}(\underline{L}_1)}{\operatorname{lev}(\underline{L}_2)}\right) \\ &\leq v_p(\det(\underline{L}_2)) - v_p(\operatorname{lev}(\underline{L}_2)) + 2v_p(\det(\sigma)) \end{aligned}$$

otherwise. When $\operatorname{rk}(\underline{L}_1) = \operatorname{rk}(\underline{L}_2) = 1$, the above bounds imply that $v_p\left(\frac{\operatorname{lev}(\underline{L}_1)}{\operatorname{lev}(\underline{L}_2)}\right) = 2v_p(\det(\sigma))$ for all primes p, in other words that $\operatorname{lev}(\underline{L}_1) = \det(\sigma)^2 \operatorname{lev}(\underline{L}_2)$. However, when $\operatorname{rk}(\underline{L}_1) = \operatorname{rk}(\underline{L}_2) > 1$, it is possible that $\operatorname{lev}(\underline{L}_1)$ differs from $\operatorname{lev}(\underline{L}_2)$ by a factor which is not a square, as illustrated by the following example:

Example 4.9. Consider the positive-definite, even lattices

$$\begin{aligned} \underline{L}_1 &= \left(\mathbb{Z}^2, \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)\right) \mapsto 8xs + 16yt\right) \text{ and} \\ \underline{L}_2 &= \left(\mathbb{Z}^2, \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)\right) \mapsto 8xs + 4yt\right). \end{aligned}$$

There exists an isometry σ_{2y} of $\underline{L}_1$ into $\underline{L}_2$, mapping $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} x \\ 2y \end{smallmatrix}\right)$. It gives rise to the linear operator $U(\sigma_{2y})$ mapping $J_{k,\underline{L}_2}$ to $J_{k,\underline{L}_1}$. Using Sage, one can check that $\operatorname{lev}(\underline{L}_1) = 32$ and $\operatorname{lev}(\underline{L}_2) = 16$, which implies that $U(\sigma_{2y})$ raises the level of the index of Jacobi forms in $J_{k,\underline{L}_2}$ by a factor of two.

If, on the other hand, we consider the isometry σ_{2x} of $\underline{L}_1$ into the lattice

$$\underline{L}_3 = \left(\mathbb{Z}^2, \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)\right) \mapsto 2xs + 16yt\right),$$

mapping $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} 2x \\ y \end{smallmatrix}\right)$, then the linear operator $U(\sigma_{2x})$ maps $J_{k,\underline{L}_3}$ to $J_{k,\underline{L}_1}$. Since $\operatorname{lev}(\underline{L}_1) = \operatorname{lev}(\underline{L}_3) = 32$, it follows $U(\sigma_{2x})$ leaves level of the index of Jacobi forms in $J_{k,\underline{L}_3}$ unchanged.

Given a positive-definite, even lattice $\underline{L}$, we want to classify lattices $\underline{L}'$ of the same rank as $\underline{L}$ such that there exists an isometry σ of $\underline{L}$ into $\underline{L}'$, since every Jacobi form in $J_{k,\underline{L}'}$ gives rise to an "oldform" in $J_{k,\underline{L}}$. For example, when $\operatorname{rk}(\underline{L}) = \operatorname{rk}(\underline{L}') = 1$, we have $\underline{L} = \underline{L}_m$ and $\underline{L}' = \underline{L}_n$ for some m and n in $\mathbb{N}$. Then the matrix of every isometry σ of $\underline{L}$ into $\underline{L}'$ is an integer d and, since $\beta' \circ \sigma(x) = \beta(x)$ for every x in $\mathbb{Z}$, we obtain that $nd^2 = m$. Hence, there is a one-to-one correspondence between lattices $\underline{L}$ such that there exists some isometry σ of $\underline{L}$ into $\underline{L}'$ and square divisors of m (this has been established in [**EZ85**]). Note that σ is unique for each $\underline{L}'$, up to multiplication by ± 1 (which is equivalent to a change of basis in $\underline{L}$ or $\underline{L}'$).

Suppose that $\underline{L} = (L,\beta)$ and $\underline{L}' = (L',\beta')$ are positive-definite, even lattices such that there exists an isometry σ of $\underline{L}$ into $\underline{L}'$. Then $(\sigma(L),\beta')$ is a sublattice of $\underline{L}'$ and

$\sigma : \underline{L} \to (\sigma(L), \beta)$ is an isomorphism of lattices. Conversely, every sublattice (M, β') of $\underline{L}'$ gives rise to an isometry of (M, β') into $\underline{L}'$ given by inclusion of (M, β') in $\underline{L}'$. Hence, the problem of classifying lattices $\underline{L}'$ of the same rank as $\underline{L}$ such that there exists an isometry σ of $\underline{L}$ into $\underline{L}'$ is equivalent to the problem of classifying overlattices of $\underline{L}$.

Proposition 4.10 ([**Nik80**, Prop 1.4.1]). *Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$. Then there is a one-to-one correspondence between overlattices of $\underline{L}$ and isotropic subgroups of $D_{\underline{L}}$. For every such overlattice $\underline{L}' = (L', \beta)$, the correspondence is given by*

$$\underline{L}' \mapsto L'/L.$$

Since we are interested in the reverse correspondence, we include the proof:

Proof. The following inclusions hold:

$$L \hookrightarrow L' \hookrightarrow L'^{\#} \hookrightarrow L^{\#}.$$

Since L has finite index in L', the group L'/L is indeed an isotropic subgroup of $L^{\#}/L$. Conversely, every isotropic subgroup I of $L^{\#}/L$ gives rise to a positive-definite lattice $\underline{L_I} = (L_I, \beta)$ containing $\underline{L}$, where L_I is the pre-image of I under the quotient map $L \to L^{\#}/L$. Since $\beta(I, L) \in \mathbb{Z}$ and

$$\beta(x, y) = \beta(x + y) - \beta(x) - \beta(y) \in \mathbb{Z}$$

for every x and y in I, it follows that $\underline{L_I}$ is even. □

If $\iota : \underline{L}_1 \xrightarrow{\sim} \underline{L}_2$ is an isomorphism of lattices, then the map

$$\iota_* : J_{k,\underline{L}_2} \to J_{k,\underline{L}_1}, \phi \mapsto \iota_*(\phi),$$

defined by

$$\iota_*(\phi)(\tau, w) = \phi(\tau, \iota(w)), \tag{4.1}$$

is an isomorphism of spaces of Jacobi forms. If $\underline{L}'$ is an overlattie of $\underline{L}$ and $\iota : \underline{L}' \xrightarrow{\sim} \underline{L}''$ is an isomorphism of lattices, then it is easy to show that $\iota \circ \sigma : \underline{L} \to \underline{L}''$ is an isometry and that $U(\sigma) \circ \iota_*(\phi) = U(\iota \circ \sigma)\phi$ for every ϕ in $J_{k,\underline{L}''}$. Two overlattices $\underline{L}'$ and $\underline{L}''$ of $\underline{L}$ are said to be *isomorphic* if there exists an automorphism of $\underline{L}$ which extends to an isomorphism between $\underline{L}'$ and $\underline{L}''$. We remind the reader that, given any positive-definite, even lattice $\underline{L}$ over $\mathbb{Z}$, every automorphism α of $\underline{L}$ extends to an automorphism $\tilde{\alpha}$ of $D_{\underline{L}}$.

Proposition 4.11 ([**Nik80**, Prop 1.4.2]). *Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$ and let $\underline{L}'$ and $\underline{L}''$ be two overlattices of $\underline{L}$. Then $\underline{L}' \simeq \underline{L}''$ if and only if there exists an automorphism α of $\underline{L}$ such that*

$$\alpha(L'/L) = L''/L.$$

We remind the reader that $\mathcal{I}_{\underline{L}}$ denotes the set of isotropic subgroups of $D_{\underline{L}}$. The orthogonal group of $\underline{L}$ acts on $\mathcal{I}_{\underline{L}}$ from the right via

$$(\alpha, I) \mapsto \tilde{\alpha}(I).$$

Let $O(\underline{L})\backslash\mathcal{I}_{\underline{L}}$ denote the quotient of this action, i.e. the set

$$\{[I] : I \in \mathcal{I}_{\underline{L}}, [I] = [J] \iff \exists\, \alpha \text{ in } O(\underline{L}) \text{ such that } J = \tilde{\alpha}(I)\}.$$

For every element I in $\mathcal{I}_{\underline{L}}$, let ι_I denote the inclusion map between L and L_I and set $U(I) := U(\iota_I)$.

DEFINITION 4.12 (Oldforms with respect to isometries). Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$. For every non-trivial isotropic subgroup I of $D_{\underline{L}}$, every element in $J_{k,\underline{L}_I}$ is called an oldform in $J_{k,\underline{L}}$. Define the space of oldfroms of weight k and index $\underline{L}$ with respect to isometries as

$$J^{\text{old, iso}}_{k,\underline{L}} := \sum_{\substack{I\in O(\underline{L})\backslash \mathcal{I}_{\underline{L}} \\ I\neq\{0\}}} U(I) J_{k,\underline{L}_I}.$$

EXAMPLE 4.13. In the case of the scalar lattice $\underline{L}_m = (\mathbb{Z}, (x, y) \mapsto 2mxy)$ ($m \in \mathbb{N}$), the isotropy set of $\underline{L}_m$ is $\mathrm{Iso}(D_{\underline{L}_m}) = \frac{1}{b}\mathbb{Z}/\mathbb{Z}$, where $m = ab^2$ with a square-free. Since $\mathrm{Iso}(D_{\underline{L}_m})$ is cyclic, all its subgroups are cyclic by the Fundamental Theorem of Cyclic Groups and they are in one to one correspondence with divisors of b. In other words,

$$\mathcal{I}_{\underline{L}_m} = \Big\{\langle s\rangle : s \in \mathrm{Iso}(D_{\underline{L}_m})\Big\} = \left\{\langle \frac{1}{d}\rangle : d \mid b\right\}$$

Note that $\underline{L}_{\langle s\rangle} = \left(\frac{1}{N_s}\mathbb{Z}, (x, y) \mapsto 2mxy\right)$ for every s in $\mathrm{Iso}(D_{\underline{L}_m})$ and the latter is isomorphic to the lattice $\underline{L}_{\frac{m}{N_s^2}} = \left(\mathbb{Z}, (x, y) \mapsto 2\frac{m}{N_s^2}xy\right)$. Furthermore, we have $O(\underline{L}_m) = \{\pm\mathrm{Id}\}$ and hence $O(\underline{L})\backslash\mathcal{I}_{\underline{L}_m} = \mathcal{I}_{\underline{L}_m}$. Thus, we recover the usual notion of oldforms with respect to the operators U_l from [**EZ85**].

EXAMPLE 4.14. When n is odd, the root lattice D_n is the maximal even lattice in the odd unimodular lattice $\mathbb{Z}^n$. In other words, it has no even overlattices. Indeed, we remind the reader that

$$D_n^\#/D_n = \left\{0, e_n, \frac{e_1 + \cdots + e_n}{2}, \frac{e_1 + \cdots + e_{n-1} - e_n}{2}\right\},$$

where $\{e_i\}_i$ denotes the standard basis of $\mathbb{Z}^n$, and hence the only isotropic element in $D_n^\#/D_n$ is the trivial one. It follows that there are no oldforms with respect to isometries in the spaces J_{k,D_n}, for every odd n and $k > \frac{n}{2}$.

4.1.1. Connection to vector-valued modular forms. A partial newform theory for vector-valued modular forms with respect to the dual Weil representaiton was developed in [**Bru14**, §3]. In this subsection, we present some of the results from there and discuss their connection with the oldform theory developed in the previous paragraphs. Let $\underline{L} = (L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and let $\underline{M} = (M,\beta)$ be a sublattice of $\underline{L}$ of finite index. We have seen that following inclusions hold:

$$M \hookrightarrow L \hookrightarrow L^\# \hookrightarrow M^\#.$$

Furthermore, the quotient group $H := L/M$ is an isotropic subgroup of $D_{\underline{M}}$ and its orthogonal complement with respect to β is $H^\perp = L^\#/M$. There is a natural isomorphism $(H^\perp/H, \beta \bmod \mathbb{Z}) \simeq D_{\underline{L}}$ and $|M^\#/M| = |L^\#/L||H|^2$.

THEOREM 4.15 ([**Sch15**, Thm 4.1]). *Let*

$$F(\tau) = \sum_{x\in L^\#/L} F_x(\tau)\mathfrak{e}_x$$

be an element of $M_k(\rho^*_{\underline{L}})$. *Then the function*

$$F_{\underline{M}}(\tau) := \sum_{x\in L^\#/M} F_{x+L/M}(\tau)\mathfrak{e}_x$$

is an element of $M_k(\rho^*_{\underline{M}})$.

While the author only treats the case where $\mathrm{rk}(\underline{L})$ is even in [**Sch15**], the proof carries through in general. We include it here:

Proof. It suffices to check that $F_{\underline{M}}$ is $\rho^*_{\underline{M}}$-invariant under the $|_k$-action of $\tilde{T}$ and $\tilde{S}$. Since $F \in M_k(\rho^*_{\underline{L}})$, the following holds for every $\tilde{A}$ in $\tilde{\Gamma}$:

$$F_x(A\tau) = w(\tau)^{2k} \sum_{y \in L\#/L} \overline{\rho_{\underline{L}}(\tilde{A})_{x,y}} F_y(\tau).$$

Since

$$\rho_{\underline{M}}(\tilde{T})\mathfrak{e}_x = \rho_{\underline{L}}(\tilde{T})\mathfrak{e}_x = e(\beta(x))\mathfrak{e}_x$$

for every x in $L^\#/M$, we have

$$F_{\underline{M}}|_k\tilde{T}(\tau) = \sum_{x \in L^\#/M} F_{x+L/M}(T\tau)\mathfrak{e}_x = \sum_{x \in L^\#/M} e(-\beta(x+L/M))F_{x+L/M}(\tau)\mathfrak{e}_x = \rho^*_{\underline{M}}(\tilde{T})F_{\underline{M}}(\tau).$$

Furthermore, write $F_{\underline{M}} = \sum_{y \in M^\#/M} F_{\underline{M},y}\mathfrak{e}_y$, with

$$F_{\underline{M},y} = \begin{cases} F_{y+L/M}, & \text{if } y \in L^\#/M \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$F_{\underline{M}}|_k\tilde{S}(\tau) = \tau^{-k} \sum_{y \in L^\#/M} F_{y+L/M}(S\tau)\mathfrak{e}_y = \tau^{-k} \sum_{y \in M^\#/M} F_{\underline{M},y}(S\tau)\mathfrak{e}_y.$$

If $y \in L^\#/M$, then

$$\begin{aligned}
\tau^{-k}F_{\underline{M},y}(S\tau) &= \tau^{-k}F_{y+L/M}(S\tau) = \sum_{x \in L^\#/L} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}} e\left(-\beta(x, y+L/M)\right) F_x(\tau) \\
&= \sum_{x \in L^\#/L} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}} |L/M| e\left(-\beta(x,y)\right) F_x(\tau) \\
&= \sum_{x \in L^\#/M} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}} e\left(-\beta(x,y)\right) F_x(\tau) \\
&= \sum_{x \in M^\#/M} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}} e\left(-\beta(x,y)\right) F_{\underline{M},x}(\tau)
\end{aligned}$$

If $y \notin L^\#/M$, then $\tau^{-k}F_{\underline{M},y}(S\tau) = 0$ and

$$\begin{aligned}
&\sum_{x \in M^\#/M} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}} e\left(-\beta(x,y)\right) F_{\underline{M},x}(\tau) = \sum_{x \in L^\#/M} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}} e\left(-\beta(x,y)\right) F_{\underline{M},x} \\
&= \sum_{x \in L^\#/L} \sum_{\mu \in L/M} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}} e\left(-\beta(x+\mu,y)\right) F_{\underline{M},x+\mu}(\tau) \\
&= \sum_{x \in L^\#/L} \frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}} e\left(-\beta(x,y)\right) F_{x+L/M}(\tau) \sum_{\mu \in L/M} e(-\beta(\mu,y)) = 0,
\end{aligned}$$

since the inner sum in the last line is equal to zero. It follows that

$$\begin{aligned} F_{\underline{M}}|_k\tilde{S}(\tau) &= \sum_{y\in M^\#/M}\left(\sum_{x\in M^\#/M}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}}e\left(-\beta(x,y)\right)F_{\underline{M},x}(\tau)\right)\mathfrak{e}_y \\ &= \sum_{x\in L^\#/M}F_{x+L/M}(\tau)\sum_{y\in M^\#/M}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}}e\left(-\beta(x,y)\right)\mathfrak{e}_y = \rho^*_{\underline{M}}(\tilde{S})F_{\underline{M}}(\tau) \end{aligned}$$

and the proof is complete. □

Given a positive-definite, even lattice $\underline{M}=(M,\beta)$, elements of $M_k(\rho^*_{\underline{M}})$ which arise in this way from overlattices of $\underline{M}$ are called *oldforms*. The converse result is the following:

Proposition 4.16 ([**Bru14**, Prop 3.2]). *Let $\underline{M}$ be a sublattice of $\underline{L}$ of finite index and let*

$$F(\tau)=\sum_{x\in M^\#/M}F_x(\tau)\mathfrak{e}_x$$

be an element of $M_k(\rho^*_{\underline{M}})$. Then the function

$$F^{\underline{L}}(\tau):=\sum_{x\in L^\#/M}F_x(\tau)\mathfrak{e}_{x+L/M}$$

is an element of $M_k(\rho^*_{\underline{L}})$.

Since an explicit proof is not given in [**Bru14**], we include it here:

Proof. The fact that

$$\rho_{\underline{L}}(\tilde{T})\mathfrak{e}_{x+L/M}=e(\beta(x))\mathfrak{e}_{x+L/M}$$

for every x in $L^\#/L$ implies that

$$F^{\underline{L}}|_k\tilde{T}(\tau)=\sum_{x\in L^\#/M}F_x(T\tau)\mathfrak{e}_{x+L/M}=\sum_{x\in L^\#/M}e(-\beta(x))F_x(\tau)\mathfrak{e}_{x+L/M}=\rho^*_{\underline{L}}(\tilde{T})F^{\underline{L}}(\tau).$$

Furthermore,

$$\begin{aligned} F^{\underline{L}}|_k\tilde{S}(\tau) &=\tau^{-k}\sum_{y\in L^\#/M}F_y(S\tau)\mathfrak{e}_{y+L/M}= \\ &=\sum_{y\in L^\#/M}\sum_{x\in M^\#/M}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|M^\#/M|}}e\left(-\beta(x,y)\right)F_x(\tau)\mathfrak{e}_{y+L/M} \\ &=\sum_{x\in M^\#/M}F_x(\tau)\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}}\frac{1}{|L/M|}\sum_{y\in L^\#/L}\sum_{\mu\in L/M}e\left(-\beta(x,y+\mu)\right)\mathfrak{e}_y \\ &=\sum_{x\in M^\#/M}F_x(\tau)\sum_{y\in L^\#/L}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}}e(-\beta(x,y))\mathfrak{e}_y\frac{1}{|L/M|}\sum_{\mu\in L/M}e(-\beta(x,\mu)) \\ &=\sum_{x\in L^\#/M}F_x(\tau)\sum_{y\in L^\#/L}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{|L^\#/L|}}e\left(-\beta(x+L/M,y)\right)\mathfrak{e}_y=\rho^*_{\underline{L}}(\tilde{S})F^{\underline{L}}(\tau). \end{aligned}$$

□

Proposition 4.17 ([**Bru14**, Prop 3.3]). *Let*

$$F(\tau)=\sum_{x\in M^\#/M}F_x(\tau)\mathfrak{e}_x$$

be an element of $M_k(\rho^*_{\underline{M}})$ *such that* $F_x = 0$ *for* x *not in* $L^\#/M$. *Then* $F_{x+y} = F_x$ *for all* y *in* L/M *and*

$$F(\tau) = \frac{1}{|L/M|}(F^{\underline{L}})_{\underline{M}}.$$

In other words, this is a sufficient criterion for an element of $M_k(\rho^*_{\underline{M}})$ *to be an oldform.*

We remind the reader of the isomorphism φ between Jacobi forms of lattice index and vector-valued modular forms for the dual Weil representation from Theorem 1.39. The connection between this newform theory for vector-valued modular forms for the dual of the Weil representation and the newform theory developed in the previous subsection for Jacobi forms of lattice index is the the following:

Lemma 4.18. *Let* $\underline{L} = (L,\beta)$ *be a positive-definite, even lattice over* $\mathbb{Z}$ *and let* $\underline{L}' = (L',\beta)$ *be an overlattice of* $\underline{L}$. *For every Jacobi form* ϕ *in* $J_{k,\underline{L}'}$, *the following holds:*

$$\varphi\left(U(L'/L)\phi\right)(\tau) = \varphi(\phi)_{\underline{L}}(\tau).$$

Proof. Since $\varphi : J_{k,\underline{L}'} \xrightarrow{\sim} M_{k-\frac{\mathrm{rk}(\underline{L})}{2}}(\rho^*_{\underline{L}'})$, it suffices to check that

$$U(L'/L)\phi(\tau,z) = \varphi^{-1}\left(\varphi(\phi)_{\underline{L}}\right)(\tau,z).$$

The Fourier expansion of the left-hand side is given in Theorem 4.3:

$$U(L'/L)\phi(\tau,z) = \sum_{\substack{r\in L'^\#, D\in\mathbb{Q}_{\le 0}\\ D\equiv\beta(r) \bmod \mathbb{Z}}} C_\phi(D,r)e\left((\beta(r)-D)\tau+\beta(r,z)\right).$$

Suppose that ϕ has the following theta expansion:

$$\phi(\tau,z) = \sum_{x\in L'^\#/L'} h_{\phi,x}(\tau)\vartheta_{\underline{L}',x}(\tau,z).$$

Then

$$\varphi(\phi)(\tau) = \sum_{x\in L'^\#/L'} h_{\phi,x}(\tau)\mathfrak{e}_x$$

and

$$\varphi(\phi)_{\underline{L}}(\tau) = \sum_{x\in L'^\#/L} h_{\phi,x+L'/L}(\tau)\mathfrak{e}_x.$$

It follows that

$$\varphi^{-1}\left(\varphi(\phi)_{\underline{L}}\right)(\tau,z) = \sum_{x\in L'^\#/L} h_{\phi,x+L'/L}(\tau)\vartheta_{\underline{L},x}(\tau,z),$$

in other words that

$$\varphi^{-1}\left(\varphi(\phi)_{\underline{L}}\right)(\tau,z) = \sum_{\substack{x\in L'^\#, D\in\mathbb{Q}_{\le 0}\\ D\equiv\beta'(x) \bmod \mathbb{Z}}} C_\phi(D,x)e\left(\beta(x)-D)\tau+\beta(x,z)\right)$$

and equality holds. □

Proposition 4.17 gives us a criterion for when a Jacobi form in $J_{k,\underline{L}}$ is an oldform with respect to isometries:

Lemma 4.19. *Let* $\underline{L}$ *be a positive-definite, even lattice over* $\mathbb{Z}$. *If*

$$\phi(\tau,z) = \sum_{(D,r)\in\mathrm{supp}(\underline{L})} C_\phi(D,r)e((\beta(r)-D)\tau+\beta(r,z))$$

is an element of $J_{k,\underline{L}}$ *such that* $C_\phi(D,r) = 0$ *for all* r *not in* $L'^\#$ *for some overlattice* $\underline{L}'$ *of* $\underline{L}$, *then* ϕ *is an oldform coming from* $J_{k,\underline{L}'}$.

Proof. Let

$$\phi(\tau,z)=\sum_{\substack{(D,r')\in\mathrm{supp}(\underline{L})\\ r'\in L'^{\#}}} C_\phi(D,r')e((\beta(r')-D)\tau+\beta(r',z))$$

be the Fourier expansion of ϕ and let

$$\phi(\tau,z)=\sum_{x\in L^{\#}/L} h_{\phi,x}(\tau)\vartheta_{\underline{L},x}(\tau,z)$$

be its theta expansion. It follows that $h_{\phi,x}=0$ for x not in $L'^{\#}/L$. First, we show that $C_\phi(D,x)=C_\phi(D,r+x)$ for every r in L'. We remind the reader that

$$h_{\phi,x}(A\tau)=w(\tau)^{2k}\sum_{y\in L^{\#}/L}\overline{\rho_{\underline{L}}(\tilde{A})_{x,y}}h_{\phi,y}(\tau)$$

for every $\tilde{A}$ in $\tilde{\Gamma}$ and therefore

$$h_{\phi,x}(S\tau)=\tau^k\sum_{y\in L^{\#}/L}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{L^{\#}/L}}e(-\beta(x,y))h_{\phi,y}(\tau)=\tau^k\sum_{y\in L'^{\#}/L}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{L^{\#}/L}}e(-\beta(x,y))h_{\phi,y}(\tau).$$

Hence, for every r in L', we have

$$\begin{aligned}h_{\phi,r+x}(S\tau)=&\tau^k\sum_{y\in L'^{\#}/L}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{L^{\#}/L}}e(-\beta(r+x,y))h_{\phi,y}(\tau)\\ =&\tau^k\sum_{y\in L'^{\#}/L}\frac{i^{-\frac{\mathrm{rk}(\underline{L})}{2}}}{\sqrt{L^{\#}/L}}e(-\beta(x,y))h_{\phi,y}(\tau)=h_{\phi,x}(S\tau).\end{aligned}$$

It follows that $h_{\phi,r+x}=h_{\phi,x}$ for every x in $L^{\#}/L'$ and every r in L', which is equivalent to the fact that $C_\phi(D,x)=C_\phi(D,r+x)$ for every r in L'.

Define

$$\phi^{\underline{L}'}(\tau,z):=\sum_{(D,x)\in\mathrm{supp}(\underline{L}')}\sum_{r\in L'/L}C_\phi(D,r+x)e((\beta'(x)-D)\tau+\beta'(x,z)).$$

Then

$$\phi^{\underline{L}'}(\tau,z)=\varphi^{-1}\left(\varphi(\phi)_{\underline{L}'}\right)(\tau,z)$$

and it follows from Proposition 4.16 that $\phi^{\underline{L}'}\in J_{k,\underline{L}'}$. But

$$\phi^{\underline{L}'}(\tau,z)=|L'/L|\sum_{(D,x)\in\mathrm{supp}(\underline{L}')}C_\phi(D,x)e((\beta'(x)-D)\tau+\beta'(x,z)),$$

since $C_\phi(D,r+x)=C_\phi(D,x)$ for every r in L', and therefore

$$U(L'/L)\phi^{\underline{L}'}(\tau,z)=|L'/L|\sum_{\substack{(D,r')\in\mathrm{supp}(\underline{L})\\ r'\in L'^{\#}}}C_\phi(D,r')e((\beta(r')-D)\tau+\beta(r',z)).$$

Thus,

$$U(L'/L)\phi^{\underline{L}'}(\tau,z)=|L'/L|\phi(\tau,z)$$

and in particular ϕ is an oldform coming from $J_{k,\underline{L}'}$. ◻

4.1.2. Theta series and level raising operators. We study the action of the operators $U(\cdot)$ on theta series.

LEMMA 4.20. *Let $\underline{L} = (L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and let $\underline{L}' = (L',\beta)$ be an overlattice of $\underline{L}$. For every r in $D_{\underline{L}'}$, the following holds:*

$$U(L'/L)\vartheta_{\underline{L}',r} = \sum_{\substack{s\in L'^{\#}/L \\ s\equiv r \bmod L'}} \vartheta_{\underline{L},s}.$$

PROOF. We have

$$\vartheta_{\underline{L}',r}(\tau,z) = \sum_{\substack{x\in L'^{\#} \\ x\equiv r \bmod L'}} e\left(\beta(x)\tau + \beta(x,z)\right)$$

by definition. Every x in $L'^{\#}$ which is congruent to r modulo L' can be written as $x = r+t$ for some t in L' and we can write $t = \lambda + \mu$, with λ in L and μ in L'/L. It follows that $x = (r+\mu) + \lambda$ and $x \in L^{\#}$ such that $x \equiv r+\mu \bmod L$ and $r+\mu \in L'^{\#}/L$ such that $r+\mu \equiv r \bmod L'$. Conversely, every x in $L'^{\#}$ which is equivalent to s modulo L for some s in $L'^{\#}/L$ which is equivalent to r modulo L' can be written as $x = s+\lambda = r+\mu+\lambda$, for some λ in L and some μ in L', and therefore $x \in L'^{\#}$ and $x \equiv r \bmod L'$. It follows that

$$U(L'/L)\vartheta_{\underline{L}',r}(\tau,z) = \sum_{\substack{s\in L'^{\#}/L \\ s\equiv r \bmod L'}} \sum_{\substack{x\in L^{\#} \\ x\equiv s \bmod L}} e\left(\beta(x)\tau + \beta(x,z)\right) = \sum_{\substack{s\in L'^{\#}/L \\ s\equiv r \bmod L'}} \vartheta_{\underline{L},s},$$

as claimed. □

Let I be an isotropic subgroup of $D_{\underline{L}}$. Since $U(I)$ is an inclusion map, we have shown that

$$\vartheta_{\underline{L}_I,r} = \sum_{\substack{s\in L_I^{\#}/L \\ s\equiv r \bmod L_I}} \vartheta_{\underline{L},s}$$

for every r in $D_{\underline{L}_I}$. We remind the reader of definition (1.17) of the $\tilde{\Gamma}$-module $\Theta_{\underline{L}}$. Define the following averaging operator on $\Theta_{\underline{L}}$:

$$\operatorname{Tr}{}^{L_I}_{\underline{L}} : \theta \mapsto \frac{1}{|L_I/L|^2} \sum_{(\lambda,\mu)\in(L_I/L)^2} \theta|_{\underline{L}}(\lambda,\mu).$$

Lemma 4.20 implies that $\Theta_{\underline{L}_I}$ is a $\tilde{\Gamma}$-submodule of $\Theta_{\underline{L}}$. Let $\Theta_{\underline{L}_I}^{\perp}$ denote the orthogonal complement of $\Theta_{\underline{L}_I}$ inside $\Theta_{\underline{L}}$ with respect to the scalar product (1.22). The following holds:

PROPOSITION 4.21. *For every θ in $\Theta_{\underline{L}}$, we have*

$$\operatorname{Tr}{}^{L_I}_{\underline{L}}\theta = 0 \iff \theta \in \Theta_{\underline{L}_I}^{\perp}.$$

Proof. First, note that $\operatorname{Tr}\frac{L_I}{\underline{L}}$ is well-defined. For every r in $L^\#/L$ and every pair (λ,μ) in $(L^\#)^2$, we have

$$\begin{aligned}
\vartheta_{\underline{L},r}|_{\underline{L}}(\lambda,\mu)(\tau,z) &= e\left(\tau\beta(\lambda)+\beta(\lambda,z)\right)\vartheta_{\underline{L},r}(\tau,z+\lambda\tau+\mu)\\
&= e\left(\tau\beta(\lambda)+\beta(\lambda,z)\right)\sum_{\substack{x\in L^\#\\ x\equiv r \bmod L}} e\left(\tau\beta(x)+\beta(x,z+\lambda\tau+\mu)\right)\\
&= \sum_{\substack{x\in L^\#\\ x\equiv r \bmod L}} e\left(\tau(\beta(x)+\beta(\lambda)+\beta(x,\lambda))+\beta(x+\lambda,z)+\beta(x,\mu)\right)\\
&= e(\beta(r,\mu))\sum_{\substack{y\in L^\#\\ y\equiv r+\lambda \bmod L}} e\left(\tau\beta(y)+\beta(y,z)\right)\\
&= e(\beta(r,\mu))\vartheta_{\underline{L},r+\lambda}(\tau,z).
\end{aligned}$$

Write $\theta=\sum_{r\in L^\#/L}c_r\vartheta_{\underline{L},r}$, with c_r in $\mathbb{C}$ for all r in $L^\#/L$. It follows that, if $(\lambda',\mu')=(\lambda,\mu)+(\delta,\gamma)$ for some (δ,γ) in L^2, then

$$\begin{aligned}
\theta|_{\underline{L}}(\lambda',\mu')(\tau,z) &= \sum_{r\in L^\#/L}c_r e(\beta(r,\mu'))\vartheta_{\underline{L},r+\lambda'}(\tau,z)=\sum_{r\in L^\#/L}c_r e(\beta(r,\mu))\vartheta_{\underline{L},r+\lambda}(\tau,z)\\
&= \theta|_{\underline{L}}(\lambda,\mu)(\tau,z)
\end{aligned}$$

and therefore $\operatorname{Tr}\frac{L_I}{\underline{L}}$ is independent of the choice of coset representatives.

Furthermore, we have

$$\begin{aligned}
\operatorname{Tr}\frac{L_I}{\underline{L}}\theta(\tau,z) &= \frac{1}{|L_I/L|^2}\sum_{r\in L^\#/L}\sum_{(\lambda,\mu)\in(L_I/L)^2}c_r e(\beta(r,\mu))\vartheta_{\underline{L},r+\lambda}(\tau,z)\\
&= \frac{1}{|L_I/L|^2}\sum_{s\in L^\#/L}\sum_{(\lambda,\mu)\in(L_I/L)^2}c_{s-\lambda}e(\beta(s,\mu))\vartheta_{\underline{L},s}(\tau,z).
\end{aligned}$$

Thus, for every $\theta_1=\sum_{r\in L^\#/L}c_r\vartheta_{\underline{L},r}$ and $\theta_2=\sum_{r\in L^\#/L}d_r\vartheta_{\underline{L},r}$ in $\Theta_{\underline{L}}$,

$$\begin{aligned}
[\operatorname{Tr}\tfrac{L_I}{\underline{L}}\theta_1,\theta_2] &= \frac{1}{|L_I/L|^2}\sum_{s\in L^\#/L}\sum_{(\lambda,\mu)\in(L_I/L)^2}c_{s-\lambda}e(\beta(s,\mu))\overline{d_s}\\
&= \frac{1}{|L_I/L|^2}\sum_{r\in L^\#/L}\sum_{(\delta,\gamma)\in(L_I/L)^2}c_r\overline{e(\beta(r,\gamma))d_{r-\delta}}\\
&= [\theta_1,\operatorname{Tr}\tfrac{L_I}{\underline{L}}\theta_2],
\end{aligned}$$

where we have made the substitutions $-\lambda=\delta$, $s=r-\delta$ and $\mu=-\gamma$. It follows that $\operatorname{Tr}\frac{L_I}{\underline{L}}$ is Hermitian with respect to the scalar product (1.22).

For every t in $D_{\underline{L}_I}$, Lemma 4.20 implies that

$$\begin{aligned}
\operatorname{Tr}\frac{L_I}{\underline{L}}\vartheta_{\underline{L}_I,t} &= \frac{1}{|L_I/L|^2}\sum_{(\lambda,\mu)\in(L_I/L)^2}\sum_{\substack{s\in L_I^\#/L\\ s\equiv t \bmod L_I}}e(\beta(s,\mu))\vartheta_{\underline{L},s+\lambda}(\tau,z)=\sum_{\lambda\in L_I/L}\sum_{\substack{x\in L_I^\#/L\\ x\equiv t \bmod L_I}}\vartheta_{\underline{L},x}(\tau,z)\\
&= \vartheta_{\underline{L}_I,t},
\end{aligned}$$

where we have used the fact that $e(\beta(s,\mu))\in\mathbb{Z}$ for every s in $L_I^\#$ and every μ in L_I and we have made the substitution $s+\lambda=x$. In other words, we have $\Theta_{\underline{L}_I}\subseteq\operatorname{Tr}\frac{L_I}{\underline{L}}\Theta_{\underline{L}}$. On the other hand, for every r in $D_{\underline{L}}$ we have

$$\operatorname{Tr}\frac{L_I}{\underline{L}}\vartheta_{\underline{L},r} = \frac{1}{|L_I/L|^2}\sum_{\lambda\in L_I/L}\vartheta_{\underline{L},r+\lambda}(\tau,z)\sum_{\mu\in L_I/L}e(\beta(r,\mu))$$

and note that the inner sum is equal to zero unless $r \in L_I^\#$. For every r in $L_I^\#$, we obtain that

$$\operatorname{Tr}\frac{L_I}{\underline{L}}\vartheta_{\underline{L},r} = \frac{1}{|L_I/L|}\sum_{\substack{s\in L_I^\#/L\\ s\equiv r \bmod L_I}}\vartheta_{\underline{L},s}(\tau,z) = U(I)\vartheta_{\underline{L_I},r}(\tau,z)$$

and thus $\operatorname{Tr}\frac{L_I}{\underline{L}}\Theta_{\underline{L}} \subseteq \Theta_{\underline{L_I}}$. It follows that $\Theta_{\underline{L_I}} = \operatorname{Tr}\frac{L_I}{\underline{L}}\Theta_{\underline{L}}$.

To prove the Proposition, assume that θ in $\Theta_{\underline{L}}$ satisfies $\operatorname{Tr}\frac{L_I}{\underline{L}}\theta = 0$. Then

$$[\theta,\Theta_{\underline{L_I}}] = [\theta,\operatorname{Tr}\tfrac{L_I}{\underline{L}}\Theta_{\underline{L}}] = [\operatorname{Tr}\tfrac{L_I}{\underline{L}}\theta,\Theta_{\underline{L}}] = 0$$

and therefore $\theta \in \Theta_{\underline{L_I}}^\perp$. Conversely, assume that $\theta \in \Theta_{\underline{L_I}}^\perp$. Then

$$0 = [\theta,\Theta_{\underline{L_I}}] = [\theta,\operatorname{Tr}\tfrac{L_I}{\underline{L}}\Theta_{\underline{L}}] = [\operatorname{Tr}\tfrac{L_I}{\underline{L}}\theta,\Theta_{\underline{L}}]$$

and therefore $\operatorname{Tr}\frac{L_I}{\underline{L}}\theta = 0$, since $[\cdot,\cdot]$ is non-degenerate. □

LEMMA 4.22. *The operator* $\operatorname{Tr}\frac{L_I}{\underline{L}}$ *is a projection map from* $\Theta_{\underline{L}}$ *to* $\Theta_{\underline{L_I}}$.

PROOF. We have seen in the proof of the previous proposition that $\Theta_{\underline{L_I}} = \operatorname{Tr}\frac{L_I}{\underline{L}}\Theta_{\underline{L}}$. Thus, we only have to check that $\operatorname{Tr}\frac{L_I}{\underline{L}}\circ\operatorname{Tr}\frac{L_I}{\underline{L}} = \operatorname{Tr}\frac{L_I}{\underline{L}}$. For every r in $D_{\underline{L}}$, we have

$$\begin{aligned}
\operatorname{Tr}\tfrac{L_I}{\underline{L}}\circ\operatorname{Tr}\tfrac{L_I}{\underline{L}}\vartheta_{\underline{L},r}(\tau,z) =& \frac{1}{|L_I/L|^2}\sum_{(\lambda,\mu)\in(L_I/L)^2}e(\beta(r,\mu))\operatorname{Tr}\tfrac{L_I}{\underline{L}}\vartheta_{\underline{L},r+\lambda}(\tau,z)\\
=&\frac{1}{|L_I/L|^4}\sum_{(\lambda,\mu)\in(L_I/L)^2}\sum_{(\delta,\gamma)\in(L_I/L)^2}e(\beta(r,\mu)+\beta(r+\lambda,\gamma))\vartheta_{\underline{L},r+\lambda+\delta}(\tau,z)\\
=&\frac{1}{|L_I/L|^4}\sum_{(x,y)\in(L_I/L)^2}|L_I/L|^2e(\beta(r,y))\vartheta_{\underline{L},r+x}(\tau,z)\\
=&\operatorname{Tr}\tfrac{L_I}{\underline{L}}\vartheta_{\underline{L},r}(\tau,z),
\end{aligned}$$

where we have used the fact that $\vartheta_{\underline{L},r}|_{\underline{L}}(\lambda,\mu) = e(\beta(r,\mu))\vartheta_{\underline{L},r+\lambda}$ for every (λ,μ) in $(L^\#)^2$ and that addition by δ and by γ are automorphisms of L_I/L for every fixed δ and γ in L_I/L. □

Set

$$\Theta_{\underline{L}}^{\mathrm{old}} := \sum_{\substack{I\in\mathcal{I}_{\underline{L}}\\ I\neq\{0\}}}\Theta_{\underline{L_I}}$$

and $\Theta_{\underline{L}}^{\mathrm{new}} := \left(\Theta_{\underline{L}}^{\mathrm{old}}\right)^\perp$, where the orthogonal complement is taken with respect to the scalar product $[\cdot,\cdot]$.

LEMMA 4.23. *The space* $\Theta_{\underline{L}}^{\mathrm{new}}$ *is a* $\tilde{\Gamma}$*-submodule of* $\Theta_{\underline{L}}$ *and, furthermore,*

$$\Theta_{\underline{L}} = \sum_{I\in\mathcal{I}_{\underline{L}}}\Theta_{\underline{L_I}}^{\mathrm{new}}. \tag{4.2}$$

PROOF. The space $\Theta_{\underline{L}}^{\mathrm{new}}$ is a $\tilde{\Gamma}$-submodule of $\Theta_{\underline{L}}$, since $\Theta_{\underline{L}_I}$ is a $\tilde{\Gamma}$-submodule of $\Theta_{\underline{L}}$ for every I in $\mathcal{I}_{\underline{L}}$ and the $\tilde{\Gamma}$-action on $\Theta_{\underline{L}}$ is unitary with respect to $[\cdot,\cdot]$ (the latter is essentially a re-statement of the fact that the Weil representation is unitary with respect to the scalar product on $\mathbb{C}[L^\#/L]$).

The statement regarding the decomposition of $\Theta_{\underline{L}}$ can be proved by induction on the number of isotropic subgroups of $D_{\underline{L}}$. If $D_{\underline{L}}$ has no non-trivial isotopic subgroups, then $\Theta_{\underline{L}} = \Theta_{\underline{L}}^{\mathrm{new}}$. For the induction step, suppose that

$$\Theta_{\underline{L}_I} = \sum_{J\in\mathcal{I}_{\underline{L}_I}} \Theta_{\underline{L}_{I+J}}^{\mathrm{new}}$$

for every non-trivial I in $\mathcal{I}_{\underline{L}}$. Then

$$\Theta_{\underline{L}_I} = \Theta_{\underline{L}}^{\mathrm{new}} \oplus \sum_{\substack{I\in\mathcal{I}_{\underline{L}}\\ I\neq\{0\}}} \sum_{J\in\mathcal{I}_{\underline{L}_I}} \Theta_{\underline{L}_{I+J}}^{\mathrm{new}} = \sum_{I\in\mathcal{I}_{\underline{L}}} \Theta_{\underline{L}_I}^{\mathrm{new}}. \qquad \square$$

PROPOSITION 4.24. *Suppose that $\underline{L}$ is a positive-definite, even lattice over $\mathbb{Z}$ such that $\beta(I,J) = 0$ for every isotropic subgroups I and J of $D_{\underline{L}}$. Then*

$$\Theta_{\underline{L}} = \bigoplus_{I\in\mathcal{I}_{\underline{L}}} \Theta_{\underline{L}_I}^{\mathrm{new}},$$

where the direct sum decomposition is taken with respect to the scalar product (1.22).

PROOF. Note that one instance in which $\beta(I,J) = 0$ for every isotropic subgroups I and J of $D_{\underline{L}}$ is when $\mathrm{Iso}(D_{\underline{L}})$ is a cyclic group (as is the case for the scalar lattices $\underline{L}_m$). In view of the previous lemma, it suffices to prove that the summands in (4.2) are pairwise orthogonal. Let I and J be two elements of $\mathcal{I}_{\underline{L}}$ such that $I \neq J$. For every r in $D_{\underline{L}_J}$ and every pair (λ,μ) in L_I^2, we have

$$\begin{aligned}
\vartheta_{\underline{L}_J,r}|_{\underline{L}}(\lambda,\mu)(\tau,z) &= e\left(\tau\beta(\lambda)+\beta(\lambda,z)\right) \sum_{\substack{x\in L_J^\#\\ x\equiv r \bmod L_J}} e\left(\tau\beta(x)+\beta(x,z+\lambda\tau+\mu)\right)\\
&= e(\beta(r,\mu)) \sum_{\substack{y\in L_J^\#\\ y\equiv r+\lambda \bmod L_J}} e\left(\tau\beta(y)+\beta(y,z)\right)\\
&= e(\beta(r,\mu))\vartheta_{\underline{L}_J,r+\lambda}(\tau,z),
\end{aligned}$$

using the fact that $\beta(I,J) = 0$. Hence,

$$\begin{aligned}
\mathrm{Tr}\,{}^{L_I}_{\underline{L}}\vartheta_{\underline{L}_J,r}(\tau,z) &= \frac{1}{|L_I/L|^2} \sum_{(\lambda,\mu)\in(L_I/L)^2} e(\beta(r,\mu))\vartheta_{\underline{L}_J,r+\lambda}(\tau,z)\\
&= \frac{1}{|L_I/L_{I\cap J}|^2}|L_{I\cap J}/L|^2 \sum_{(\delta,\gamma)\in(L_I/L_{I\cap J})^2} \sum_{(s,t)\in(L_{I\cap J}/L)^2} e(\beta(r,\gamma+t))\vartheta_{\underline{L}_J,r+\delta+s}(\tau,z)\\
&= \frac{1}{|L_I/L_{I\cap J}|^2} \sum_{(\delta,\gamma)\in(L_I/L_{I\cap J})^2} e(\beta(r,\gamma))\vartheta_{\underline{L}_J,r+\delta}(\tau,z).
\end{aligned}$$

The quotient $I' := I/I\cap J$ is a non-trivial isotropic subgroup of $D_{\underline{L}_J}$ and we have $L_I/L_{I\cap J} = (L_I + J/I\cap J)/(L_{I\cap J} + J/I\cap J) = L_{J+I'}/L_J$. It follows that

$$\mathrm{Tr}\,{}^{L_I}_{\underline{L}}\vartheta_{\underline{L}_J,r}(\tau,z) = \mathrm{Tr}^{L_{J+I'}}_{\underline{L}_J}\vartheta_{\underline{L}_J,r}(\tau,z).$$

Proposition 4.21 implies that the latter is equal to zero if $\vartheta_{\underline{L}_J,r} \in \Theta_{\underline{L}_J}^{\mathrm{new}}$ and, implicitly, that $\Theta_{\underline{L}_J}^{\mathrm{new}} \subseteq \Theta_{\underline{L}_I}^{\perp}$. Since $\Theta_{\underline{L}_I}^{\perp} \subseteq (\Theta_{\underline{L}_I}^{\mathrm{new}})^{\perp}$, the proof is complete. $\square$

4.2. The V operators

Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$. For every l in $\mathbb{N}$, consider the linear operator $U(lL/L) : J_{k,\underline{L}} \to J_{k,(lL,\beta)}$ arising from the following inclusion of lattices:

$$(lL, \beta) \hookrightarrow \underline{L}.$$

The map $\iota : (L, l^2\beta) \to (lL, \beta)$ defined by $x \mapsto lx$ is an isomorphism of lattices. Define a linear operator $U(l)$ on the space $J_{k,\underline{L}}$ as the composition of $U(lL/L)$ with the map ι_* defined in (4.1), in other words

$$U(l)\phi(\tau, z) := \iota_* \left(U(lL/L)\phi\right)(\tau, z) = \phi(\tau, lz).$$

Equivalently, the operator $U(l)$ is the operator $U(\sigma_l)$ corresponding to the isometry

$$\sigma_l : (L, l^2\beta) \mapsto (L, \beta), \quad , \sigma_l(x) = lx.$$

Hence, it maps $J_{k,\underline{L}}$ to $J_{k,\underline{L}(l^2)}$ and, if ϕ in $J_{k,\underline{L}}$ has a Fourier expansion of the type

$$\phi(\tau, z) = \sum_{\substack{n \in \mathbb{Z}, r \in L^\# \\ n \geq \beta(r)}} c_\phi(n, r) e(n\tau + \beta(r, z)),$$

then Theorem 4.3 implies that $U(l)\phi$ has the following Fourier expansion:

$$U(l)\phi(\tau, z) = \sum_{\substack{n \in \mathbb{Z}, r' \in L(l^2)^\# \\ n \geq l^2\beta(r')}} c_\phi(n, lr') e(n\tau + l^2\beta(r', z)),$$

with the convention that $c_\phi(n, lr') = 0$ unless r' is an l-th multiple of another element of $L(l^2)^\#$. The level of $\underline{L}(l^2)$ is equal to $l^2 \operatorname{lev}(\underline{L})$ and that the determinant of $\underline{L}(l^2)$ is equal to $l^{2\operatorname{rk}(\underline{L})} \det(\underline{L})$.

Extend the definition of $U(l)$ to l in $\mathbb{R}_{\geq 0}$. We remind the reader of definition (3.3) of the set $\Gamma\backslash \mathrm{M}(l)$ and that the $|_{k,\underline{L}}$-action of a matrices in $\mathrm{M}_2^+(\mathbb{Z})$ on holomorphic, complex-valued functions defined on $\mathfrak{H} \times (L \otimes \mathbb{C})$ is defined in (3.4).

Definition 4.25. For every l in $\mathbb{N}$, define a linear operator $V(l)$ on the space $J_{k,\underline{L}}$ as

$$V(l)\phi(\tau, z) := l^{\frac{k}{2}-1} \sum_{M \in \Gamma\backslash \mathrm{M}(l)} U(\sqrt{l}) \left(\phi|_{k,\underline{L}} M\right)(\tau, z).$$

This definition was given in [**EZ85**] for Jacobi forms of scalar index. We remind the reader that the set

$$\Delta_l = \{A = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) : a, b, d \in \mathbb{Z}, a, d \geq 0, ad = l \text{ and } 0 \leq b < d\}$$

is as a set of coset representatives of $\Gamma\backslash \mathrm{M}(l)$. If $M = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Delta_l$, then $\phi|_{k,\underline{L}} M(\tau, z)$ contains a factor of $\phi\left(M\tau, \frac{\sqrt{l}z}{d}\right)$ and this function transforms like a Jacobi form with respect to translations in the sublattice $\sqrt{l}(\tau L \oplus L)$ in the abelian variable. However, this sublattice is incommensurable with $\tau L \oplus L$. Applying $U(\sqrt{l})$ restores integrality and brings us closer to obtaining a function which is invariant with respect to translations in $\tau L \oplus L$ in the abelian variable. In other words, the operators $V(\cdot)$ are "precursors" of the usual Hecke operators. The following holds:

Theorem 4.26. *For every l in $\mathbb{N}$, the operator $V(l)$ is independent of the choice of coset representatives of the action of* $\Gamma\backslash \mathrm{M}_2(\mathbb{Z})$. *It maps $J_{k,\underline{L}}$ to $J_{k,\underline{L}(l)}$. Furthermore, if ϕ in $J_{k,\underline{L}}$ has a Fourier expansion of the type*

$$\phi(\tau, z) = \sum_{\substack{n \in \mathbb{Z}, r \in L^\# \\ n \geq \beta(r)}} c_\phi(n, r) e(n\tau + \beta(r, z)),$$

then $V(l)\phi$ has the following Fourier expansion:

$$V(l)\phi(\tau,z)=\sum_{\substack{n\in\mathbb{Z},r'\in L(l)^\#\\ n\geq l\beta(r')}}\sum_{\substack{a|(n,l)\\ \frac{r'}{a}\in L(l)^\#}}a^{k-1}c_\phi\left(\frac{nl}{a^2},\frac{lr'}{a}\right)e(n\tau+l\beta(r',z)).\tag{4.3}$$

Proof. Fix a coset $[M]$ in $\Gamma\backslash\mathrm{M}(l)$ and assume that $[M^*]$ is a different choice of representative for the same coset. Then $M^*=\gamma M$ for some γ in Γ and hence

$$\phi|_{k,\underline{L}}M^*(\tau,z)=\phi|_{k,\underline{L}}(\gamma M)(\tau,z)=\phi|_{k,\underline{L}}\gamma|_{k,\underline{L}}M(\tau,z)=\phi|_{k,\underline{L}}M(\tau,z),$$

since $\phi\in J_{k,\underline{L}}$. It follows that $V(l)$ is independent of the choice of coset representatives.

To show that $V(l)\phi$ is invariant under the $|_{k,\underline{L}(l)}$-action of Γ, let $\{M_i\}_i$ be a set of coset representatives of $\Gamma\backslash\mathrm{M}(l)$. For every A in Γ, [**CS17**, Lemma 6.3.1] implies that $M_iA=A_iM_{\sigma(i)}$ for some permutation σ on the set of indices $\{i\}$ and some A_i in Γ. It follows that

$$\begin{aligned}\sum_i\phi|_{k,\underline{L}}M_i|_{k,\underline{L}}A(\tau,z)&=\sum_i\phi|_{k,\underline{L}}(M_iA)(\tau,z)=\sum_i\phi|_{k,\underline{L}}(A_iM_{\sigma(i)})(\tau,z)\\&=\sum_i\phi|_{k,\underline{L}}A_i|_{k,\underline{L}}M_{\sigma(i)}(\tau,z)=\sum_j\phi|_{k,\underline{L}}M_j(\tau,z),\end{aligned}$$

since $\phi\in J_{k,\underline{L}}$ and where $j=\sigma(i)$. Set

$$\psi(\tau,z):=l^{\frac{k}{2}-1}\sum_{\Gamma\backslash\mathrm{M}(l)}\phi|_{k,\underline{L}}M(\tau,z)$$

for simplicity. Then

$$\begin{aligned}V(l)\phi|_{k,\underline{L}(l)}A(\tau,z)=&(U(\sqrt{l})\psi)|_{k,\underline{L}(l)}A(\tau,z)=U(\sqrt{l})\psi\left(A\tau,\frac{z}{c\tau+d}\right)(c\tau+d)^{-k}e\left(\frac{-cl\beta(z)}{c\tau+d}\right)\\=&\psi\left(A\tau,\frac{\sqrt{l}z}{c\tau+d}\right)(c\tau+d)^{-k}e\left(\frac{-c\beta(\sqrt{l}z)}{c\tau+d}\right)=\psi|_{k,\underline{L}}A(\tau,\sqrt{l}z)=\psi(\tau,\sqrt{l}z)\\=&U(\sqrt{l})\psi(\tau,z)=V(l)\phi.\end{aligned}$$

To check for invariance under the action of $H^{\underline{L}(l)}(\mathbb{Z})$, take $\{M_i\}_i=\Delta_l$. For each $M_i=\left(\begin{smallmatrix}a&b\\0&d\end{smallmatrix}\right)$ in Δ_l, set

$$\delta_i(\tau,z):=\left(\frac{d}{\sqrt{l}}\right)^kU(\sqrt{l})(\phi|_{k,\underline{L}}M_i)(\tau,z)=\phi\left(\frac{a\tau+b}{d},az\right)$$

for simplicity. We have

$$\delta_i|_{\underline{L}(l)}(\lambda,\mu)=e(\tau l\beta(\lambda)+l\beta(\lambda,z))\phi\left(\frac{a\tau+b}{d},az+a\lambda\tau+a\mu\right).$$

Substitute τ' for $\frac{a\tau+b}{d}$ and z' for az, which implies that $\tau=\frac{d\tau'-b}{a}$ and that $z=\frac{z'}{a}$:

$$\begin{aligned}\delta_i|_{\underline{L}(l)}(\lambda,\mu)(\tau,z)=&e\left(d(\tau'd-b)\beta(\lambda)+\beta(d\lambda,z')\right)\phi\left(\tau',z'+(\tau'd-b)\lambda+a\mu\right)\\=&e\left(\tau'\beta(d\lambda)+\beta(d\lambda,z')\right)\phi\left(\tau',z'+\tau'd\lambda+(a\mu-b\lambda)\right)\\=&\phi|_{\underline{L}}(d\lambda,a\mu-b\lambda)(\tau',z')=\phi(\tau',z')=\delta_i(\tau,z),\end{aligned}$$

since $\phi\in J_{k,\underline{L}}$. It follows that $V(l)\phi|_{\underline{L}(l)}h=V(l)\phi$ for every h in $H^{\underline{L}(l)}(\mathbb{Z})$. It remains to prove that $V(l)\phi$ has the correct Fourier expansion.

Take Δ_l as the set of coset representatives of $\Gamma\backslash\mathrm{M}(l)$ in the definition of $V(l)$:

$$V(l)\phi(\tau,z)=\frac{1}{l}\sum_{ad=l}a^k\sum_{b\in\mathbb{Z}_{(d)}}\phi\left(\frac{a\tau+b}{d},az\right).\tag{4.4}$$

Insert the Fourier expansion of ϕ in order to obtain that

$$V(l)\phi(\tau,z) = \frac{1}{l}\sum_{ad=l} a^k \sum_{b \bmod d} \sum_{\substack{n\in\mathbb{Z}, r\in L^\# \\ n\geq\beta(r)}} c_\phi(n,r) e\left(\frac{na}{d}\tau + \beta(r,az)\right) e_d(nb).$$

Since

$$\frac{1}{d}\sum_{b(d)} e_d(nb) = \begin{cases} 1, & \text{if } d \mid n \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

substitute m for $n\frac{a}{l}$ in the above:

$$V(l)\phi(\tau,z) = \sum_{a\mid l} a^{k-1} \sum_{\substack{m\in\mathbb{Z}, r\in L^\# \\ m\geq\frac{a}{l}\beta(r)}} c_\phi\left(\frac{ml}{a}, r\right) e(ma\tau + a\beta(r,z)).$$

Substitute n for ma, which implies that the condition on a becomes $a \mid (n,l)$ and that the condition on n and r becomes $n \geq \frac{a^2}{l}\beta(r)$. Furthermore, set $ar = s$:

$$V(l)\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, s\in L^\# \\ n\geq\frac{1}{l}\beta(s)}} \sum_{\substack{a\mid(n,l) \\ \frac{s}{a}\in L^\#}} a^{k-1} c_\phi\left(\frac{nl}{a^2}, \frac{s}{a}\right) e(n\tau + \beta(s,z)),$$

with the usual convention that an empty sum is equal to zero. There is a one-to-one correspondence between $L(l)^\#$ and $L^\#$, given by $x \mapsto lx$. Set $\frac{s}{l} = r'$ in order to complete the proof. □

Corollary 4.27. *Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$ and let $l \in \mathbb{N}$. Then $V(l)$ maps $S_{k,\underline{L}}$ to $S_{k,\underline{L}(l)}$.*

Proof. If ϕ in $S_{k,\underline{L}}$ and has a Fourier expansion of the type

$$\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r\in L^\# \\ n>\beta(r)}} c_\phi(n,r) e(n\tau + \beta(r,z)),$$

then the above theorem implies that $V(l)\phi$ has the following Fourier expansion:

$$V(l)\phi(\tau,z) = \sum_{\substack{n\in\mathbb{Z}, r'\in L(l)^\# \\ n\geq l\beta(r')}} \sum_{\substack{a\mid(n,l) \\ \frac{r'}{a}\in L(l)^\#}} a^{k-1} c_\phi\left(\frac{nl}{a^2}, \frac{lr'}{a}\right) e(n\tau + l\beta(r',z)).$$

If $n = l\beta(r')$ and a satisfies the conditions in the above equation, then $\frac{nl}{a^2} = \beta\left(\frac{lr'}{a}\right)$ and therefore $c_\phi\left(\frac{nl}{a^2}, \frac{lr'}{a}\right) = 0$, since ϕ is a cusp form. It follows that $V(\sigma)\phi$ is also a cusp form. □

We will show that the $V(\cdot)$ operators preserve Eisenstein series in the following sections.

The level of $\underline{L}(l)$ is equal to $l\,\mathrm{lev}(\underline{L})$ and its determinant is equal to $l^{\mathrm{rk}(\underline{L})}\det(\underline{L})$. It follows that $V(l)$ also *raises the level* of the index of Jacobi forms that it is applied to.

Remark 4.28. Jacobi forms of lattice index can be obtained as the Fourier–Jacobi coefficients of orthogonal modular forms. Orthogonal modular forms have many applications in algebraic geometry. For example, they can be the automorphic discriminants of moduli spaces [**GN98**], which allows for the construction of modular varieties [**Gri18**]. The operators $V(\cdot)$ were constructed in [**Gri94**] as the images of the elliptic Hecke operators (1.7) under a certain homomorphism, using the embedding of spaces of Jacobi forms into spaces of orthogonal modular forms.

Given a positive-definite, even lattice $\underline{L} = (L,\beta)$ and a positive integer l, the lattice $\underline{L}(1/l)$ is even if and only if $l \mid \gcd(G)$ and $\frac{1}{l}G$ has even diagonal entries, where G is any Gram matrix of β (note that change of basis preserves the greatest common divisor of the Gram matrix). When that is the case, we say that l divides β and write $l \mid \beta$. For example, when $\mathrm{rk}(\underline{L}) = 1$, we have $\underline{L} = \underline{L}_m$ for some m in $\mathbb{N}$ and $\underline{L}(1/l)$ is even if and only if $l \mid m$.

Definition 4.29 (Oldforms with respect to $V(\cdot)$ operators). Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$. For every positive integer $l \mid \beta$ which is greater than one and every ϕ in $J_{k,\underline{L}(1/l)}$, the Jacobi form $V(l)\phi$ is an oldform in $J_{k,\underline{L}}$. Define the space of oldfroms of weight k and index $\underline{L}$ with respect to the $V(\cdot)$ operators as

$$J_{k,\underline{L}}^{\text{old, V}} := \sum_{\substack{l \| \beta \\ l>1}} V(l) J_{k,\underline{L}(1/l)}.$$

Example 4.30. We remind the reader of definition (1.24) of the scalar Eisenstein series. Theorem 4.3 in [**EZ85**] states that, if m is a square-free, positive integer, then

$$V(m)E_{k,\underline{L}_1,0} = \sigma_{k-1}(m)E_{k,\underline{L}_m,0},$$

in other words $E_{k,\underline{L}_m,0}$ is an oldform.

Example 4.31. The root lattice D_1 has Gram matrix equal to 2 with respect to the standard basis element of $\mathbb{Z}$. For $n > 1$, a $\mathbb{Z}$-basis of D_n is given by the set

$$\{e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1}, e_1 + e_2\}.$$

The Gram matrix of the Euclidean bilinear form with respect to this basis is equal to $\left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$ when $n = 2$ and to

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & 0 & \ldots & 0 & 2 \end{pmatrix}$$

when $n > 2$. It follows that there are no oldforms with respect to the $V(\cdot)$ operators in the spaces J_{k,D_n}, for every $n > 2$ and $k \geq \frac{n}{2}$. We have also seen in Example 4.14 that there are no oldforms with respect to isometries in the spaces J_{k,D_n} when n is odd. Nonetheless, Table 3.1 illustrates that there are Jacobi forms in the spaces J_{8,D_3}, J_{10,D_7}, J_{10,D_3}, J_{12,D_7}, J_{12,D_5} and J_{12,D_3} which might lift to old elliptic modular forms.

4.3. Properties

We establish the commutative properties of $U(\cdot)$ and $V(\cdot)$ and their combined action on Eisenstein series.

Lemma 4.32. *Let $\underline{L} = (L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and suppose that I and J are two isotropic subgroups of $D_{\underline{L}}$. Then $I + J$ is an isotropic subgroup of $D_{\underline{L}}$ if and only if $\beta(I,J) = 0$ and, when this is the case,*

$$U(J) \circ U(I) = U(I) \circ U(J) = U(I + J).$$

Proof. Every element x in $I + J$ can be written as $x = r + s$ with r in I and s in J. It follows that $\beta(x) = \beta(r) + \beta(s) + \beta(r, s) = \beta(r, s)$ and therefore $I + J$ is an isotropic subgroup of $D_{\underline{L}}$ if and only if $\beta(r, s) = 0$ for every r in I and s in J. Furthermore, the following inclusions hold when $\beta(I, J) = 0$:

$$\underline{L} \hookrightarrow \underline{L}_I \hookrightarrow \underline{L}_{I+J}$$
$$\underline{L} \hookrightarrow \underline{L}_J \hookrightarrow \underline{L}_{J+I}.$$

Since the $U(\cdot)$ operators are inclusion maps, the result follows. □

LEMMA 4.33. *Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and suppose that I is an isotropic subgroup of $D_{\underline{L}}$. Then I is also an isotropic subgroup of $D_{\underline{L}(l)}$ for every l in $\mathbb{N}$ and*

$$V(l) \circ U(I) = U(I) \circ V(l).$$

PROOF. We have $\beta(I) = 0 \implies l\beta(I) = 0$ and hence I is an isotropic subgroup of $D_{(L,l\beta)}$. Since $U(I)$ acts as an inclusion map, the result follows. □

LEMMA 4.34. *For every l and l' in $\mathbb{N}$, the following holds:*

$$V(l) \circ V(l')\phi = \sum_{d|(l,l')} d^{k-1} U(d) \circ V\left(\frac{ll'}{d^2}\right)\phi. \tag{4.5}$$

PROOF. Analyse the Fourier expansions of both sides of the above equation. Equation (4.3) implies that the Fourier coefficient of $e(n\tau + \beta(r,z))$ on the left-hand side of (4.5) is equal to

$$\sum_{\substack{b|(n,l)\\ \frac{r}{b}\in L^\#}} b^{k-1} \sum_{\substack{a|\left(\frac{nl}{b^2},l'\right)\\ \frac{r}{ab}\in L^\#}} a^{k-1} c_\phi\left(\frac{nll'}{a^2b^2}, \frac{ll'r}{ab}\right) = \sum_e N(e) e^{k-1} c_\phi\left(\frac{nll'}{e^2}, \frac{ll'r}{e}\right), \tag{4.6}$$

where $N(e)$ is equal to the number of ways of writing $e = ab$ with the conditions in the sums. In order to make these conditions precise, write

$$n = t_1 b, \tag{4.7}$$

$$l = t_2 b, \tag{4.8}$$

$$\frac{nl}{b^2} = t_1 t_2 = t_3 a \text{ and} \tag{4.9}$$

$$l' = t_4 a, \tag{4.10}$$

with t_1, t_2, t_3 and t_4 in $\mathbb{N}$. Equation (4.10) implies that

$$t_4 e = l'b \iff \frac{t_4 e}{(l',e)} = \frac{l'b}{(l',e)}.$$

But $\left(\frac{e}{(l',e)}, \frac{l'}{(l',e)}\right) = 1$ and therefore $\frac{l'}{(l',e)} \mid t_4$. This implies that $b = \frac{e}{(l',e)}\delta$, with $\delta = t_4(l',e)/l'$. Since $b \mid e, b \mid n$ and $b \mid l$, it follows that δ divides e, n and l as well. Since δ divides t_4, it also divides l'. Equation (4.9) implies that

$$\frac{nl}{e} = \frac{nl}{b^2} \times \frac{b}{a} = \frac{t_3 ab}{a} = t_3 b$$

and hence $\delta \mid (nl/e)$. Combining equations (4.7) and (4.10), we obtain that $nl'/e = t_1 t_4$ and thus $\delta \mid (nl'/e)$. Combining equations (4.8) and (4.10), we obtain that $ll'/e = t_2 t_4$ and therefore $\delta \mid (ll'/e)$. Finally, equations (4.9) and (4.10) imply that $nll'/e^2 = t_3 t_4 a$ and hence $\delta \mid (nll'/e^2)$. We obtain that

$$\delta \mid \left(n, l, l', e, \frac{nl}{e}, \frac{nl'}{e}, \frac{ll'}{e}, \frac{nll'}{e^2}\right). \tag{4.11}$$

In the converse direction, we want to show that the conditions in the above equation imply the conditions in the sums in (4.6). Suppose that $e \mid (nl, nl', ll')$, that $e^2 \mid nll'$ and that δ satisfies (4.11). Write

$$\begin{aligned} n &= s_1\delta, & l &= s_2\delta, \\ l' &= s_3\delta, & e &= s_4\delta, \\ \tfrac{nl}{e} &= s_5\delta = \delta\frac{s_1s_2}{s_4}, & \frac{nl'}{e} &= s_6\delta = \delta\frac{s_1s_3}{s_4}, \\ \tfrac{ll'}{e} &= \delta s_7 = \delta\frac{s_2s_3}{s_4} \text{ and} & \frac{nll'}{e^2} &= \delta s_8 = \delta\frac{s_1s_2s_3}{s_4^2}. \end{aligned}$$

Set $b := \frac{e}{(l',e)}\delta$ and $a := \frac{e}{b} = \frac{(l',e)}{\delta}$. It follows that $a = (\delta s_3, \delta s_4)/\delta = (s_3, s_4)$ and, in particular, that $a \mid l'$. To show that $b \mid n$, we need to show that $\frac{e}{(l',e)} \mid s_1$. But $e/(l',e) = s_4/(s_3,s_4)$ and $s_1s_3/s_4 = s_6$ is an integer, which implies that $\frac{s_1s_3/(s_3,s_4)}{s_4/(s_3,s_4)} \in \mathbb{Z}$. Since $\left(\frac{s_3}{(s_3,s_4)}, \frac{s_4}{(s_3,s_4)}\right) = 1$, it follows that $\frac{s_4}{(s_3,s_4)} \mid s_1$ and implicitly that $b \mid n$. To show that $b \mid l$, we need to show that $\frac{e}{(l',e)} \mid s_2$ or, equivalently, that $\frac{s_4}{(s_3,s_4)} \mid s_2$. Since $s_2s_3/s_4 = s_7$ is an integer, it follows that $\frac{s_2s_3/(s_3,s_4)}{s_4/(s_3,s_4)} \in \mathbb{Z}$. Hence, $\frac{s_4}{(s_3,s_4)} \mid s_2$ and implicitly $b \mid l$. Finally, we need to show that $a \mid \frac{nl}{b^2}$. We have

$$\frac{nl}{b^2} = \frac{s_1s_2}{e^2/(e,l')^2} = \frac{s_1s_2(s_3,s_4)^2}{s_4^2} = \frac{s_5(s_3,s_4)}{s_4/(s_3,s_4)}$$

and $a = (s_3, s_4)$. Thus, we need to show that $\frac{s_4}{(s_3,s_4)} \mid s_5$. We know that

$$s_8 = \frac{s_1s_2s_3}{s_4^2} = \frac{s_5s_3}{s_4} = \frac{s_5s_3/(s_3,s_4)}{s_4/(s_3,s_4)}.$$

Since $\left(\frac{s_3}{(s_3,s_4)}, \frac{s_4}{(s_3,s_4)}\right) = 1$, we have $\frac{s_4}{(s_3,s_4)} \mid s_5$ and therefore $a \mid \frac{nl}{b^2}$. Thus,

$$N(e) = \#\left\{\delta : \delta \mid \left(n, l, l', e, \frac{nl}{e}, \frac{nl'}{e}, \frac{ll'}{e}, \frac{nll'}{e^2}\right)\right\},$$

with the added condition that $\frac{r}{e} \in L^\#$.

On the other hand, the coefficient of $e(n\tau + \beta(r,z))$ on the right-hand side of (4.5) is equal to

$$\sum_{d|(l,l')} d^{k-1} \sum_{\substack{a\left(n,\frac{ll'}{d^2}\right) \\ \frac{r}{ad}\in L^\#}} a^{k-1} c_\phi\left(\frac{ll'n}{a^2d^2}, \frac{ll'r}{ad}\right) = \sum_e N'(e) e^{k-1} c_\phi\left(\frac{ll'n}{e^2}, \frac{ll'r}{e}\right),$$

where $N'(e)$ is equal to the number of ways of writing $e = ad$ with the conditions in the sums. Following the same argument as above, write

$$\begin{aligned} l &= t_5 d, & l' &= t_6 d, \\ n &= t_7 a \text{ and} & \frac{ll'}{d^2} &= t_5t_6 = t_8 a. \end{aligned}$$

Write $d = \frac{e}{(n,e)}\delta$ with $\delta = t_7(n,e)/e$ and obtain that δ divides e, l, l' and n. Furthermore, we have $ll'/e = t_8 d$, $ln/e = t_5t_7$, $l'n/e = t_6t_7$ and $ll'n/e^2 = t_8t_7$. It follows that $N'(e) = N(e)$ and the proof is complete. ∎

Corollary 4.35. *If $\phi \in J_{k,\underline{L}}$ and $(l,l') = 1$, then*

$$V(l') \circ V(l)\phi = V(ll')\phi.$$

Theorem 4.36. *The operators $U(\cdot)$ and $V(\cdot)$ commute with Hecke operators. In other words, let $\underline{L} = (L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and let I be an isotropic subgroup of $D_{\underline{L}}$. If $\phi \in J_{k,\underline{L}_I}$ and $l \in \mathbb{N}_{\underline{L}} \cap \mathbb{N}_{\underline{L}_I}$, then*

$$\text{(i)} \qquad T(l) \circ U(I)\phi = U(I) \circ T(l)\phi.$$

If $\phi \in J_{k,\underline{L}}$, $l \in \mathbb{N}_{\underline{L}}$ and $l' \in \mathbb{N}$ such that $(l,l') = 1$, then

$$\text{(ii)} \qquad T(l) \circ V(l')\phi = V(l') \circ T(l)\phi.$$

Proof. Since Hecke operators are defined by different formulas for even and odd rank lattices, respectively, consider the case where $\mathrm{rk}(\underline{L})$ is odd for their commutativity with $U(\cdot)$ and the case where $\mathrm{rk}(\underline{L})$ is even for their commutativity with $V(\cdot)$. The remaining two cases can be treated analogously.

(i) We remind the reader of the action of Hecke operators on the Fourier coefficients of a Jacobi form of odd rank lattice index, as stated in Theorem 3.5: if ϕ has Fourier expansion

$$\phi(\tau,z) = \sum_{(D,r)\in \mathrm{supp}(\underline{L}_I)} C_\phi(D,r) e((\beta(r) - D)\tau + \beta(r,z)),$$

then

$$T(l)\phi(\tau,z) = \sum_{(D,r)\in \mathrm{supp}(\underline{L}_I)} C_{T(l)\phi}(D,r) e((\beta(r) - D)\tau + \beta(r,z)),$$

where

$$C_{T(l)\phi}(D,r) = \sum_{\substack{a\mid l^2 \\ a^2 \mid l^2 \,\mathrm{lev}(\underline{L}_I) D}} a^{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2}\rceil -1} \mu_{\underline{L}_I}(D,a) C_\phi\left(\frac{l^2}{a^2}D, la'r\right)$$

and a' is an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L}_I)$. Furthermore, Theorem 4.3 implies that

$$U(I)\phi(\tau,z) = \sum_{\substack{(D,r')\in \mathrm{supp}(\underline{L}) \\ r' \in L_I^\#}} C_\phi(D,r') e((\beta(r') - D)\tau + \beta(r',z)).$$

The Fourier expansion of $T(l) \circ U(I)\phi$ is

$$\begin{aligned} T(l) \circ U(I)\phi(\tau,z) = \sum_{(D,r')\in \mathrm{supp}(\underline{L})} \; \sum_{\substack{a \mid l^2 \\ a^2 \mid l^2 \,\mathrm{lev}(\underline{L}) D, la'r' \in L_I^\#}} & a^{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2}\rceil -1} \mu_{\underline{L}}(D,a) C_\phi\left(\frac{l^2}{a^2}D, la'r'\right) \\ & \times e((\beta(r') - D)\tau + \beta(r',z)), \end{aligned}$$

where a' is an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L})$. On the other hand,

$$\begin{aligned} U(I) \circ T(l)\phi(\tau,z) = \sum_{\substack{(D,r')\in \mathrm{supp}(\underline{L}) \\ r' \in L_I^\#}} \; \sum_{\substack{A \mid l^2 \\ A^2 \mid l^2 \,\mathrm{lev}(\underline{L}_I) D}} & A^{k-\lceil \frac{\mathrm{rk}(\underline{L})}{2}\rceil -1} \mu_{\underline{L}_I}(D,A) C_\phi\left(\frac{l^2}{A^2}D, lA'r'\right) \\ & \times e((\beta(r') - D)\tau + \beta(r',z)), \end{aligned}$$

where A' is an integer such that $AA' \equiv 1 \bmod \mathrm{lev}(\underline{L}_I)$.

Compare conditions in the summations first. Since $\mathrm{lev}(\underline{L}_I) \mid \mathrm{lev}(\underline{L})$ and $l \in \mathbb{N}_{\underline{L}} \cap \mathbb{N}_{\underline{L}_I}$, if a divides l^2, then $a^2 \mid l^2 \,\mathrm{lev}(\underline{L})D$ if and only if $a^2 \mid l^2 \,\mathrm{lev}(\underline{L}_I)D$. Furthermore, clearly $r' \in L_I^\# \implies lA'r' \in L_I^\#$. Conversely, since $l \in \mathbb{N}_{\underline{L}_I}$ and $AA' \equiv 1 \bmod \mathrm{lev}(\underline{L}_I)$, we have $lA'r' = r^*$ for some r^* in $L_I^\#$. This is equivalent to the fact that $r' \equiv l^{-1}ar^* \bmod L$, where l^{-1} denotes the inverse of l modulo $\mathrm{lev}(\underline{L}_I)$. Hence, $r' \in L_I^\# \iff lA'r' \in L_I^\#$ and therefore the conditions in the summations are equivalent to one another.

We remind the reader that

$$\mu_{\underline{L}}(D,A) = \begin{cases} \chi_{\underline{L}}\left(\frac{D}{f^2}, \frac{A}{f^2}\right), & \text{if } (\mathrm{lev}(\underline{L})D, A) = f^2 \text{ for some } f \text{ in } \mathbb{N} \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

and that

$$\chi_{\underline{L}}\left(\frac{D}{f^2}, \frac{A}{f^2}\right) = \left(\frac{(D/f^2)(-1)^{\lfloor \frac{\mathrm{rk}(\underline{L})}{2} \rfloor} 2\det(\underline{L})}{A/f^2}\right).$$

Since $A \mid l^2$ and $l \in \mathbb{N}_{\underline{L}} \cap \mathbb{N}_{\underline{L}_l}$, if follows that

$$(\mathrm{lev}(\underline{L})D, A) = (\mathrm{lev}(\underline{L}_l)D, A).$$

We have

$$\chi_{\underline{L}_l}\left(\frac{D}{f^2}, \frac{A}{f^2}\right) = \left(\frac{(D/f^2)(-1)^{\lfloor \frac{\mathrm{rk}(\underline{L})}{2} \rfloor} 2\det(\underline{L}_l)}{A/f^2}\right) = \chi_{\underline{L}}\left(\frac{D}{f^2}, \frac{A}{f^2}\right),$$

since $\frac{\det(\underline{L})}{\det(\underline{L}_l)}$ is a square by Lemma 4.8 and it is coprime to A as a result of Remark 1.8. Last, but not least, if a' is an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L})$ and A' is an integer such that $aA' \equiv 1 \bmod (\mathrm{lev}(\underline{L}_l))$, then it is straight-forward to show that $a' = A' + m\,\mathrm{lev}(\underline{L}_l)$ for some integer m. It follows from Remark 1.8 that $la'r' \equiv lA'r' \bmod L_l$ and hence

$$C_\phi\left(\frac{l^2}{a^2}D, la'r'\right) = C_\phi\left(\frac{l^2}{a^2}D, lA'r'\right).$$

(ii) We remind the reader of the action of Hecke operators on the Fourier coefficients of a Jacobi form of even rank lattice index, as stated in Theorem 3.6: if ϕ has Fourier expansion

$$\phi(\tau,z) = \sum_{(D,r)\in\mathrm{supp}(\underline{L})} C_\phi(D,r)e((\beta(r)-D)\tau + \beta(r,z)),$$

then

$$T(l)\phi(\tau,z) = \sum_{(D,r)\in\mathrm{supp}(\underline{L})} C_{T(l)\phi}(D,r)e((\beta(r)-D)\tau + \beta(r,z)),$$

where

$$C_{T(l)\phi}(D,r) = \sum_{a\mid l^2, \mathrm{lev}(\underline{L})D} a^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}\chi_{\underline{L}}(a)C_\phi\left(\frac{l^2}{a^2}D, la'r\right)$$

and a' is an integer such that $aa' \equiv 1 \bmod \mathrm{lev}(\underline{L})$. Furthermore, Theorem 4.26 implies that

$$\begin{aligned} V(l')\phi(\tau,z) &= \sum_{(D,r')\in\mathrm{supp}(L(l'))} \sum_{\substack{a\mid(l'\beta(r')-D), l' \\ \frac{r'}{a}\in L(l')^\#}} a^{k-1}c_\phi\left(\frac{(l'\beta(r')-D)l'}{a^2}, \frac{l'r'}{a}\right) \\ &\quad \times e\left((l'\beta(r')-D)\tau + l'\beta(r',z)\right) \\ &= \sum_{(D,r')\in\mathrm{supp}(L(l'))} \sum_{\substack{a\mid(l'\beta(r')-D), l' \\ \frac{r'}{a}\in L(l')^\#}} a^{k-1}C_\phi\left(\frac{Dl'}{a^2}, \frac{l'r'}{a}\right)e\left((l'\beta(r')-D)\tau + l'\beta(r',z)\right), \end{aligned}$$

where $C_\phi(D,r) = c_\phi(l'\beta(r)-D, r)$, as usual. The Fourier expansion of $T(l)\circ V(l')\phi$ is

$$\begin{aligned} T(l)\circ V(l')\phi(\tau,z) = &\sum_{(D,r')\in\mathrm{supp}(\underline{L}(l'))} \sum_{b\mid l^2, l'\,\mathrm{lev}(\underline{L})D} b^{k-\frac{\mathrm{rk}(\underline{L})}{2}-1} \sum_{\substack{a\mid l^2(b'^2 l'\beta(r')-\frac{D}{b^2}), l' \\ \frac{lb'r'}{a}\in L(l')^\#}} a^{k-1}\chi_{\underline{L}(l')}(b) \\ &\times C_\phi\left(\frac{l^2Dl'}{b^2a^2}, \frac{l'lb'r'}{a}\right)e((l'\beta(r')-D)\tau + l'\beta(r',z)), \end{aligned}$$

where b' is an integer such that $bb' \equiv 1 \bmod \operatorname{lev}(\underline{L}(l'))$. On the other hand,

$$V(l') \circ T(l)\phi(\tau, z) = \sum_{(D,r') \in \operatorname{supp}(\underline{L}(l''))} \sum_{\substack{A \mid (l'\beta(r')-D), l' \\ \frac{r'}{A} \in L(l')^{\#}}} A^{k-1} \sum_{B \mid l^2, \operatorname{lev}(\underline{L})D} B^{k-\frac{\operatorname{rk}(\underline{L})}{2}-1} \chi_{\underline{L}}(B)$$
$$\times C_\phi\left(\frac{l^2 D l'}{B^2 A^2}, \frac{lB'l'r'}{A}\right) e\left((l'\beta(r') - D)\tau + l'\beta(r', z)\right),$$

where B' is an integer such that $BB' \equiv 1 \bmod \operatorname{lev}(\underline{L})$.

Compare conditions in the summations first. Since b and B divide l^2 and $(l', l) = 1$, they parametrize the same sets. Since $(l, l') = 1$, if $b \mid l^2$ and $a \mid l'$, then

$$a \mid l^2\left(b'^2 l'\beta(r') - \frac{D}{b^2}\right) \iff a \mid \frac{l^2}{b}\left(bb'^2 l'\beta(r') - \frac{D}{b}\right)$$
$$\iff a \mid \left(bb'^2 l'\beta(r') - \frac{D}{b}\right) \iff a \mid \left(b^2 b'^2 l'\beta(r') - D\right).$$

A similar argument as in (i) implies that $r' \in AL(l'^2)^{\#} \iff lB'r' \in AL(l')^{\#}$ and therefore the conditions in the summations are equivalent to one another.

We remind the reader that

$$\chi_{\underline{L}}(B) = \left(\frac{(-1)^{\frac{\operatorname{rk}(\underline{L})}{2}} \det(\underline{L})}{B}\right)$$

and that

$$\chi_{\underline{L}(l')}(B) = \left(\frac{(-1)^{\frac{\operatorname{rk}(\underline{L}(l'))}{2}} l'^{\operatorname{rk}(\underline{L})} \det(\underline{L})}{B}\right).$$

Since $\operatorname{rk}(\underline{L})$ is even, it follows that $\left(\frac{l'^{\operatorname{rk}(\underline{L})}}{B}\right) = 1$ and therefore equality holds between the two above quantities. As before, if b' is an integer such that $bb' \equiv 1 \bmod \operatorname{lev}(\underline{L}(l'))$ and B' is an integer such that $bB' \equiv 1 \bmod (\operatorname{lev}(\underline{L}))$, then

$$C_\phi\left(\frac{l^2 D l'}{b^2 a^2}, \frac{l' l b' r'}{a}\right) = C_\phi\left(\frac{l^2 D l'}{b^2 a^2}, \frac{lB'l'r'}{a}\right),$$

completing the proof. □

THEOREM 4.37. *Let $\underline{L}$ be a positive-definite, even lattice over $\mathbb{Z}$ and let I be an isotropic subgroup of $D_{\underline{L}}$. If $\phi \in J_{k,\underline{L}_I}$ and s is an element of $O(D_{\underline{L}}) \cap O(D_{\underline{L}_I})$, then*

(i) $$U(I)(\phi W(s)) = (U(I)\phi)W(s).$$

If $\phi \in J_{k,\underline{L}}$, $l \in \mathbb{N}$ and s is an element of $O(D_{\underline{L}(l)}) \cap O(D_{\underline{L}})$, then

(ii) $$V(l)(\phi W(s)) = (V(l)\phi)W(s).$$

PROOF. (i) We remind the reader that $L_I^{\#}/L_I$ is a subgroup of $L^{\#}/L$. Since $U(I)$ is an inclusion map of $J_{k,\underline{L}_I}$ into $J_{k,\underline{L}}$, the result follows.

(ii) Let s be an element of $O(D_{\underline{L}(l)}) \cap O(D_{\underline{L}})$. Equations(3.12) and (4.3) imply that

$$V(l)(\phi W(s))(\tau, z) = V(l)\Big(\sum_{\substack{n \in \mathbb{Z}, r \in L^{\#} \\ n \geq \beta(r)}} c_\phi(n, s(r)) e(n\tau + \beta(r, z))\Big)$$
$$= \sum_{\substack{n \in \mathbb{Z}, r' \in L(l)^{\#} \\ n \geq l\beta(r')}} \sum_{\substack{a \mid (n,l) \\ \frac{r'}{a} \in L(l)^{\#}}} a^{k-1} c_\phi\left(\frac{nl}{a^2}, s\left(\frac{lr'}{a}\right)\right) e(n\tau + l\beta(r', z))$$

and that

$$(V(l)\phi)W(s)(\tau,z) = \Bigg(\sum_{\substack{n\in\mathbb{Z},r'\in L(l)^\#\\ n\geq l\beta(r')}}\Bigg(\sum_{\substack{a|(n,l)\\ \frac{r'}{a}\in L(l)^\#}} a^{k-1}c_\phi\left(\frac{nl}{a^2},\frac{lr'}{a}\right)\Bigg)e(n\tau+l\beta(r',z))\Bigg)W(s)$$

$$= \sum_{\substack{n\in\mathbb{Z},r'\in L(l)^\#\\ n\geq l\beta(r')}}\sum_{\substack{a|(n,l)\\ \frac{r'}{a}\in L(l)^\#}} a^{k-1}c_\phi\left(\frac{nl}{a^2},\frac{l}{a}s(r')\right)e(n\tau+l\beta(r',z))$$

and equality holds between these two expressions. □

Finally, we study the action of $U(\cdot)$ and $V(\cdot)$ on Eisenstein series.

Proposition 4.38. *Let $\underline{L}=(L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and let I be an isotropic subgroup of $D_{\underline{L}}$. For every r in $\mathrm{Iso}(D_{\underline{L}_I})$, the following holds:*

$$U(I)E_{k,\underline{L}_I,r} = \sum_{\substack{s\in L_I^\#/L\\ s\equiv r \bmod L_I}} E_{k,\underline{L},s}. \tag{4.12}$$

Proof. Since Jacobi Eisenstein series are uniquely determined by their singular term (2.17) and since the theta series $\{\vartheta_{\underline{L},r} : r\in\mathrm{Iso}(D_{\underline{L}})\}$ are linearly independent as functions of z, it suffices to prove that equality holds in (4.12) for the singular terms. For every s in $L_I^\#/L$ such that $s\equiv r \bmod L_I$, we have $\beta(s)=\beta(r+\lambda)\in\mathbb{Z}$ for some λ in L_I. Therefore, the right-hand side is well-defined and its singular term is equal to

$$\sum_{\substack{s\in L_I^\#/L\\ s\equiv r \bmod L_I}} \frac{1}{2}\left(\vartheta_{\underline{L},s}(\tau,z)+(-1)^k\vartheta_{\underline{L},-s}(\tau,z)\right).$$

The singular term of the left-hand side is equal to

$$\frac{1}{2}\left(\vartheta_{\underline{L}_I,r}(\tau,z)+(-1)^k\vartheta_{\underline{L}_I,-r}(\tau,z)\right)$$

and Lemma 4.20 implies that equality holds between the two singular terms. □

Hence, the operators $U(\cdot)$ map Eisenstein series to Eisenstein series (in addition to preserving cusp forms). Conversely, given a positive-definite, even lattice $\underline{L}$, we want do determine which Eisenstein series are oldforms with respect to isometries (i.e. coming from overlattices of $\underline{L}$). We remind the reader of Definition 1.31 of twisted Eisenstein series. Let $x\in\mathscr{R}_{\mathrm{Iso}}$ and ξ be a primitive Dirichlet character modulo F with $F\mid N_x$. Then

$$E_{k,\underline{L},x,\xi} = \sum_{d\in\mathbb{Z}^\times_{(N_x)}} \xi(d)E_{k,\underline{L},dx}.$$

Write $N_x = N_0'N_0$ with $N_0=\prod_{p|F}p^{v_p(N_x)}$. For every divisor f of N_0', set $x_f := fFx$. Then $\langle x_f\rangle$ is an isotropic subgroup of $D_{\underline{L}}$ of order $N_{x_f}=N_0N_0'/fF$ and $\underline{L}_{\langle x_f\rangle}=(L_{\langle x_f\rangle},\beta)$ is a positive-definite, even overlattice of $\underline{L}$.

For every isotropic element r in $L^\#_{\langle x_f\rangle}/L_{\langle x_f\rangle}$, Proposition 4.38 implies that

$$U(\langle x_f\rangle)E_{k,\underline{L}_{\langle x_f\rangle},r} = \sum_{\substack{s\in L^\#_{\langle x_f\rangle}/L\\ s\equiv r \bmod L_{\langle x_f\rangle}}} E_{k,\underline{L},s}.$$

Every s in the above summation can be written as $s=r+y$ for some y in $L_{\langle x_f\rangle}$ and, since $s\in L^\#_{\langle x_f\rangle}/L$, it follows that $y\in L_{\langle x_f\rangle}/L$. Since $\lambda\neq\mu$ in $L_{\langle x_f\rangle}/L$ if and only if

$\lambda-\mu \in L+dx_f$ for some d in $\mathbb{Z}_{(N_{x_f})}$, a set of coset representatives for this quotient group is given by $\{dx_f : d \in \mathbb{Z}_{(N_{x_f})}\}$. It follows that

$$U(\langle x_f \rangle)E_{k,\underline{L}_{\langle x_f\rangle},r} = \sum_{d \in \mathbb{Z}_{(N_{x_f})}} E_{k,\underline{L},r+dx_f} \tag{4.13}$$

and note that $N_{x_f} = N_x/fF$. The following result was proved in [**Sch18**] for vector-valued modular forms:

Theorem 4.39. *If ξ is a primitive Dirichlet character modulo F for some $F \mid N_x$ such that $F \neq N_x$, then $E_{k,\underline{L},x,\xi}$ is an oldfrom. More precisely, we have*

$$E_{k,\underline{L},x,\xi}(\tau,z) = \xi(N_0')\sum_{f \mid N_0'} \mu(f)U(\langle x_f\rangle)E_{k,\underline{L}_{\langle x_f\rangle},N_0'x,\xi}(\tau,z).$$

Proof. Since N_0 and N_0' are coprime, every element d in $\mathbb{Z}^\times_{(N_x)}$ can be written as $d = mN_0 + nN_0'$, with m running through $\mathbb{Z}^\times_{(N_0')}$ and n running through $\mathbb{Z}^\times_{(N_0)}$ as d runs through $\mathbb{Z}^\times_{(N_x)}$. Since $F \mid N_0$ and they share the same set of prime divisors, it follows that n can be written as $n = a + bF$, with a running through $\mathbb{Z}^\times_{(F)}$ and b running through $\mathbb{Z}_{(N_0/F)}$ as n runs through $\mathbb{Z}^\times_{(N_0)}$. It follows that

$$\begin{aligned} E_{k,\underline{L},x,\xi} &= \sum_{m\in\mathbb{Z}^\times_{(N_0')}} \sum_{a\in\mathbb{Z}^\times_{(F)}} \sum_{b\in\mathbb{Z}_{(N_0/F)}} \xi(mN_0 + (a+bF)N_0')E_{k,\underline{L},(mN_0+(a+bF)N_0')x,\xi} \\ &= \xi(N_0')\sum_{a\in\mathbb{Z}^\times_{(F)}} \xi(a) \sum_{b\in\mathbb{Z}_{(N_0/F)}} \sum_{m\in\mathbb{Z}^\times_{(N_0')}} E_{k,\underline{L},(mN_0+(a+bF)N_0')x,\xi}. \end{aligned}$$

Remove the coprimality conditions between m and N_0' using (1.4). Set $f = (m, N_0')$ in the above equation and obtain that

$$\begin{aligned} E_{k,\underline{L},x,\xi} &= \xi(N_0')\sum_{a\in\mathbb{Z}^\times_{(F)}} \xi(a) \sum_{b\in\mathbb{Z}_{(N_0/F)}} \sum_{f\mid N_0'}\mu(f) \sum_{e\in\mathbb{Z}_{(N_0'/f)}} E_{k,\underline{L},(feN_0+(a+bF)N_0')x,\xi} \\ &= \xi(N_0')\sum_{f\mid N_0'}\mu(f) \sum_{a\in\mathbb{Z}^\times_{(F)}} \xi(a) \sum_{b\in\mathbb{Z}_{(N_0/F)}} \sum_{e\in\mathbb{Z}_{(N_0'/f)}} E_{k,\underline{L},(eN_0/F+bN_0'/f)x_f+aN_0'x,\xi}. \end{aligned}$$

The expression $eN_0/F + bN_0'/f$ runs through $N_0N_0'/fF = N_{x_f}$ as b runs through $\mathbb{Z}_{(N_0/F)}$ and e runs through $\mathbb{Z}_{(N_0'/f)}$. Since $x \in \mathrm{Iso}(D_{\underline{L}})$, we have $\beta(N_0'x, L_{\langle x_f\rangle}) \in \mathbb{Z}$ and $\beta(N_0'x) \in \mathbb{Z}$. It follows that $N_0'x$ is an isotropic element of order F in $L^\#_{\langle x_f\rangle}/L_{\langle x_f\rangle}$ and (4.13) implies that

$$\begin{aligned} E_{k,\underline{L},x,\xi} &= \xi(N_0')\sum_{f\mid N_0'}\mu(f) \sum_{a\in\mathbb{Z}^\times_{(F)}} \xi(a) \sum_{d\in\mathbb{Z}_{(N_{x_f})}} E_{k,\underline{L},dx_f+aN_0'x,\xi} \\ &= \xi(N_0')\sum_{f\mid N_0'}\mu(f) \sum_{a\in\mathbb{Z}^\times_{(N_{N_0'x})}} \xi(a)U(\langle x_f\rangle)E_{k,\underline{L}_{\langle x_f\rangle},aN_0'x} \\ &= \xi(N_0')\sum_{f\mid N_0'}\mu(f)U(\langle x_f\rangle)E_{k,\underline{L}_{\langle x_f\rangle},N_0'x,\xi}, \end{aligned}$$

as claimed. ◻

Proposition 4.40. *Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice over $\mathbb{Z}$. For every r in* $\mathrm{Iso}(D_{\underline{L}})$ *and every l in $\mathbb{N}$, the following holds:*

$$V(l)E_{k,\underline{L},r} = \sum_{s \in \mathrm{Iso}(D_{\underline{L}(l)})} \sum_{\substack{a|(l\beta(s),l) \\ r \equiv \frac{ls}{a} \bmod L}} a^{k-1} E_{k,\underline{L}(l),s}. \tag{4.14}$$

Proof. As was the case in the proof of Proposition 4.38, it suffices to prove that equality holds for the singular terms. The singular term of the right-hand side of (4.14) is equal to

$$\sum_{s \in \mathrm{Iso}(D_{\underline{L}(l)})} \sum_{\substack{a|(l\beta(s),l) \\ r \equiv \frac{ls}{a} \bmod L}} \frac{a^{k-1}}{2} \left(\vartheta_{\underline{L}(l),s} + (-1)^k \vartheta_{\underline{L}(l),-s} \right).$$

The Fourier expansion of the left-hand side is

$$\sum_{(D,x) \in \mathrm{supp}(\underline{L}(l))} \sum_{\substack{a|(l\beta(x)-D,l) \\ \frac{x}{a} \in L(l)^\#}} a^{k-1} G_{k,\underline{L},r}\left(\frac{Dl}{a^2}, \frac{lx}{a}\right) e\left((l\beta(x) - D)\tau + l\beta(x,z)\right)$$

and therefore its singular terms is equal to

$$\begin{aligned} &\sum_{\substack{x \in L(l)^\# \\ l\beta(x) \in \mathbb{Z}}} \sum_{\substack{a|(l\beta(x),l) \\ \frac{x}{a} \in L(l)^\#}} a^{k-1} G_{k,\underline{L},r}\left(0, \frac{lx}{a}\right) e\left(l\beta(x)\tau + l\beta(x,z)\right) \\ &= \sum_{\substack{s \in L(l)^\#/L \\ l\beta(s) \in \mathbb{Z}}} \sum_{\substack{a|(l\beta(s),l) \\ \frac{s}{a} \in L(l)^\#}} a^{k-1} G_{k,\underline{L},r}\left(0, \frac{ls}{a}\right) \sum_{\substack{x \in L(l)^\# \\ x \equiv s \bmod L}} e\left(l\beta(x)\tau + l\beta(x,z)\right). \end{aligned}$$

For every r and s in $L^\#$, define

$$\delta_L(r,s) := \begin{cases} 1, & \text{if } r \equiv s \bmod L \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

We remind the reader that

$$G_{k,\underline{L},r}\left(0, \frac{ls}{a}\right) = \frac{1}{2}\left(\delta_L\left(r, \frac{ls}{a}\right) + (-1)^k \delta_L\left(-r, \frac{ls}{a}\right)\right).$$

Note that, if $\frac{ls}{a} \equiv r \bmod L$ or $\frac{ls}{a} \equiv -r \bmod L$, then $\frac{s}{a} \in L(l)^\#$. Thus, the singular term of the left-hand side of (4.14) is equal to

$$\begin{aligned} &\frac{1}{2} \sum_{\substack{s \in L(l)^\#/L \\ l\beta(s) \in \mathbb{Z}}} \sum_{a|(l\beta(s),l)} a^{k-1} \delta_L\left(r, \frac{ls}{a}\right) \vartheta_{\underline{L}(l),s}(\tau,z) + \frac{(-1)^k}{2} \sum_{\substack{s \in L(l)^\#/L \\ l\beta(s) \in \mathbb{Z}}} \sum_{a|(l\beta(s),l)} a^{k-1} \delta_L\left(-r, \frac{ls}{a}\right) \vartheta_{\underline{L}(l),s}(\tau,z) \\ &= \sum_{\substack{s \in L(l)^\#/L \\ l\beta(s) \in \mathbb{Z}}} \sum_{a|(l\beta(s),l)} a^{k-1} \delta_L\left(r, \frac{ls}{a}\right) \frac{1}{2} \left(\vartheta_{\underline{L}(l),s}(\tau,z) + (-1)^k \vartheta_{\underline{L}(l),-s}(\tau,z) \right), \end{aligned}$$

after substituting $s := -s$ in the second line. Hence, equality holds between the two singular terms. ☐

In particular, Propositions 4.38 and 4.40 imply that

$$\begin{aligned} U(I) &: J^{\mathrm{Eis}}_{k,\underline{L}_I} \to J^{\mathrm{Eis}}_{k,\underline{L}} \text{ and} \\ V(l) &: J^{\mathrm{Eis}}_{k,\underline{L}} \to J^{\mathrm{Eis}}_{k,\underline{L}(l)}. \end{aligned}$$

They can be combined into the following result:

Proposition 4.41. *Let $\underline{L} = (L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and let I be an isotropic subgroup of $D_{\underline{L}}$. For every r in $\mathrm{Iso}(D_{\underline{L}_I})$ and every l in $\mathbb{N}$, the following holds:*

$$V(l) \circ U(I)E_{k,\underline{L}_I,r} = \sum_{x\in \mathrm{Iso}(D_{\underline{L}(l)})} \sum_{\substack{a|(l\beta(x),l)\\ r\equiv \frac{lx}{a} \bmod L_I}} a^{k-1}E_{k,\underline{L}(l),x}. \tag{4.15}$$

Proof. Equations (4.12) and (4.14) imply that

$$\begin{aligned} V(l) \circ U(I)E_{k,\underline{L}_I,r} &= \sum_{\substack{s\in L_I^{\#}/L\\ s\equiv r \bmod L_I}} V(l)E_{k,\underline{L},s} = \sum_{\substack{s\in L_I^{\#}/L\\ s\equiv r \bmod L_I}} \sum_{x\in\mathrm{Iso}(D_{\underline{L}(l)})} \sum_{\substack{a|(l\beta(x),l)\\ s\equiv \frac{lx}{a} \bmod L}} a^{k-1}E_{k,\underline{L}(l),x} \\ &= \sum_{x\in\mathrm{Iso}(D_{\underline{L}(l)})} \sum_{a|(l\beta(x),l)} a^{k-1}E_{k,\underline{L}(l),x} \sum_{\substack{s\in L_I^{\#}/L\\ s\equiv r \bmod L_I}} \delta_L\left(s, \frac{lx}{a}\right). \end{aligned}$$

The inner sum contains at most one non-trivial term. Suppose that x in $\mathrm{Iso}(D_{\underline{L}(l)})$ and $a \mid (l\beta(x), l)$ are fixed. If there exists an s in $L_I^{\#}/L$ such that $s \equiv r \bmod L_I$ and $\frac{lx}{a} \equiv s \bmod L$, then

$$\begin{aligned} \frac{lx}{a} \equiv s \bmod L \implies \frac{lx}{a} &= s + \mu \text{ for some } \mu \text{ in } L \\ \implies \frac{lx}{a} &= r + \lambda \text{ for some } \lambda \text{ in } L_I \implies \frac{lx}{a} \equiv r \bmod L_I. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{lx}{a} \equiv r \bmod L_I \implies \frac{lx}{a} &= r + \lambda \text{ for some } \lambda \text{ in } L_I \\ \implies \frac{lx}{a} &= r + \gamma + \mu \text{ for some } \gamma \text{ in } I \text{ and some } \mu \text{ in } L \\ \implies \frac{lx}{a} &\equiv s \bmod L \end{aligned}$$

for some s in $L_I^{\#}/L$ such that $s \equiv r \bmod L + I$. In other words, there exists an s in $L_I^{\#}/L$ such that $s \equiv r \bmod L_I$ and $\frac{lx}{a} \equiv s \bmod L$ if and only if $\frac{lx}{a} \equiv r \bmod L$. It follows that

$$\sum_{\substack{s\in L_I^{\#}/L\\ s\equiv r \bmod L_I}} \delta_L\left(s, \frac{lx}{a}\right) = \delta_{L_I}\left(r, \frac{lx}{a}\right)$$

and we obtain the desired result. □

Corollary 4.42. *Let $\underline{L} = (L,\beta)$ be a positive-definite, even lattice over $\mathbb{Z}$ and let I be an isotropic subgroup of $D_{\underline{L}}$. For every l in $\mathbb{N}$, the following holds:*

$$V(l) \circ U(I)E_{k,\underline{L}_I,0} = \sum_{\substack{x\in\left(\frac{1}{l}L_I\right)/L\\ l\beta(x)\in\mathbb{Z}}} \sum_{\substack{a|(l\beta(x),l)\\ \frac{lx}{a}\in L_I}} a^{k-1}E_{k,\underline{L}(l),x}. \tag{4.16}$$

Proof. When $r = 0$ in Proposition 4.41, the right-hand side of (4.15) vanishes identically unless $lx \in L_I$. The following equalities hold:

$$\{x \in L(l)^{\#}/L : lx \in L_I\} = \{[x] : x \in \frac{1}{l}L_I \text{ and } [x] = [s] \iff x - s \in L\} = \left(\frac{1}{l}L_I\right)/L.$$

The result follows immediately from Proposition 4.41. □

REMARK 4.43. Assume that k is odd. Then $E_{k,\underline{L}_I,0} = 0$ and, for every x in $\left(\frac{1}{l}L_I\right)/L$ such that $l\beta(x) \in \mathbb{Z}$, $-x$ satisfies the same condition. Since $E_{k,\underline{L}(l),x} = (-1)^k E_{k,\underline{L}(l),-x}$ for every x in $L(l)^{\#}$, it follows that both sides of (4.16) vanish.

The above corollary leads to the following generalization of (13) from [**EZ85**, §I.4]:

$$\begin{aligned}
&\frac{1}{l^{k-1}}\prod_{p\mid l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2\mid l}\mu(d)V\left(\frac{l}{d^2}\right)\circ U(d)E_{k,\underline{L},0}\\
&=\frac{1}{l^{k-1}}\prod_{p\mid l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2\mid l}\mu(d)\sum_{\substack{x\in\left(\frac{1}{l}L\right)/L\\ l\beta(x)\in\mathbb{Z}}}\sum_{\substack{a\mid\left(l\beta(x),\frac{l}{d^2}\right)\\ \frac{lx}{da}\in L}}a^{k-1}E_{k,\underline{L}(l),x}.
\end{aligned}\tag{4.17}$$

The unintuitive normalizing factor on the left-hand side of (4.17) was chosen because, when k is even, the coefficient corresponding to $E_{k,\underline{L}(l),0}$ on the right-hand side of this equation is equal to one. We include the proof of this claim. When $x \in L$, $l\beta(x)$ is an integer multiple of $\frac{l}{d^2}$ and $\frac{lx}{da} \in L$ for every divisor a of $\frac{l}{d^2}$ and therefore the coefficient of $E_{k,\underline{L}(l),0}$ is equal to

$$\frac{1}{l^{k-1}}\prod_{p\mid l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2\mid l}\mu(d)\sum_{a\mid\frac{l}{d^2}}a^{k-1}.$$

Define

$$F(l) := \frac{1}{l^{k-1}}\prod_{p\mid l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2\mid l}\mu(d)\sigma_{k-1}\left(\frac{l}{d^2}\right).$$

We show, by induction on the number of primes dividing l, that $F(l) = 1$ for every positive integer l. This clearly holds when $l = 1$. Assume that $l = p^\kappa$ for some prime number p and some $\kappa \geq 1$. If $\kappa = 1$, then

$$F(l) = \frac{1}{p^{k-1}} \times \frac{1}{1+p^{-(k-1)}}\sigma_{k-1}(p) = 1.$$

If $\kappa > 1$, then every divisor d of p^κ is of the form $d = p^a$ for some $0 \leq a \leq \kappa$ and $\mu(d) = 0$ for $a > 1$. Therefore,

$$\begin{aligned}
F(p^\kappa) =&\frac{1}{p^{\kappa(k-1)}} \times \frac{1}{1+p^{-(k-1)}}\left(\sigma_{k-1}(p^\kappa)-\sigma_{k-1}(p^{\kappa-2})\right)\\
=&\frac{1}{p^{\kappa(k-1)}+p^{(\kappa-1)(k-1)}} \times \frac{p^{(\kappa+1)(k-1)}-p^{(\kappa-1)(k-1)}}{p^{k-1}-1} = 1.
\end{aligned}$$

In the induction step, assume that $F(l) = 1$ whenever l is the product of t distinct primes, say $l = \prod_{p_i\mid l} p_i^{a_i}$ for some fixed number of primes t, and show that this implies that $F(lp^\kappa) = 1$, for every $p \neq p_i$ for every i in $\{1,\ldots,t\}$ and every $\kappa > 0$. If $\kappa = 1$, then $d^2 \mid lp^\kappa \iff d^2 \mid l$ and hence

$$\begin{aligned}
F(lp) =&\frac{1}{p^{k-1}} \times \frac{1}{1+p^{-(k-1)}} \times \frac{1}{l^{k-1}}\prod_{p_i\mid l}\frac{1}{1+p_i^{-(k-1)}}\sum_{d^2\mid l}\mu(d)\sigma_{k-1}\left(\frac{lp}{d^2}\right)\\
=&\frac{1}{p^{k-1}+1} \times \frac{1}{l^{k-1}}\prod_{p_i\mid l}\frac{1}{1+p_i^{-(k-1)}}\sum_{d^2\mid l}\mu(d)\sigma_{k-1}\left(\frac{l}{d^2}\right)\sigma_{k-1}(p) = 1,
\end{aligned}$$

where we have used the fact that the divisor sum is multiplicative and that $\left(p, \frac{l}{d^2}\right) = 1$ for every $d^2 \mid l$. If $\kappa > 1$, then the set of square divisors d'^2 of lp^κ that satisfy $\mu(d') \neq 0$ is equal to the union of the set of square divisors d^2 of l that satisfy $\mu(d) \neq 0$ with the set of square divisors of the form p^2d^2, with d in the former set. Note that $\mu(pd) = -\mu(d)$ for such d. Hence,

$$\begin{aligned}
F(lp^\kappa) =& \frac{1}{(lp^\kappa)^{k-1}} \prod_{p_j|(lp^\kappa)} \frac{1}{1+p_j^{-(k-1)}} \sum_{d^2|lp^\kappa} \mu(d)\sigma_{k-1}\left(\frac{lp^\kappa}{d^2}\right) \\
=& \frac{1}{p^{\kappa(k-1)}} \times \frac{1}{1+p^{-(k-1)}} \times \frac{1}{l^{k-1}} \prod_{p_i|l} \frac{1}{1+p_i^{-(k-1)}} \sum_{d^2|l} \mu(d)\left(\sigma_{k-1}\left(\frac{lp^\kappa}{d^2}\right) - \sigma_{k-1}\left(\frac{lp^\kappa}{p^2d^2}\right)\right) \\
=& \frac{1}{p^{\kappa(k-1)} + p^{(\kappa-1)(k-1)}} \times \frac{1}{l^{k-1}} \prod_{p_i|l} \frac{1}{1+p_i^{-(k-1)}} \\
& \times \sum_{d^2|l} \mu(d)\sigma_{k-1}\left(\frac{l}{d^2}\right)\left(\sigma_{k-1}\left(p^\kappa\right) - \sigma_{k-1}(p^{\kappa-2})\right) = F(p^\kappa)F(l) = 1
\end{aligned}$$

and the proof is complete. We include the proof of [**EZ85**, §I.4, (13)], since it is not given explicitly in the cited text:

LEMMA 4.44. *For every l in $\mathbb{N}$, the following holds:*

$$\frac{1}{l^{k-1}} \prod_{p|l} \frac{1}{1+p^{-(k-1)}} \sum_{d^2|l} \mu(d) V\left(\frac{l}{d^2}\right) \circ U(d) E_{k,\underline{L}_1,0} = E_{k,\underline{L}_l,0}.$$

PROOF. Take $\underline{L} = \underline{L}_1$ in (4.17). Then the right-hand side of this equation is equal to

$$\frac{1}{l^{k-1}} \prod_{p|l} \frac{1}{1+p^{-(k-1)}} \sum_{d^2|l} \mu(d) \sum_{\substack{x\in\mathbb{Z}_{(l)} \\ \frac{x^2}{l}\in\mathbb{Z}}} \sum_{\substack{a|\left(\frac{x^2}{l},\frac{l}{d^2}\right) \\ \frac{x}{da}\in\mathbb{Z}}} a^{k-1} E_{k,\underline{L}_l,\frac{x}{l}}.$$

Write l as $l = bc^2$, with b square-free. The condition that $d^2 \mid l$ is equivalent to $d \mid c$, the condition that $l \mid x^2$ is equivalent to $x = bcs$ for some $s \in \mathbb{Z}_{(c)}$ and it follows that the above expression is equal to

$$\frac{1}{l^{k-1}} \prod_{p|l} \frac{1}{1+p^{-(k-1)}} \sum_{d|c} \mu(d) \sum_{s\in\mathbb{Z}_{(c)}} \sum_{\substack{a|b\left(s,\frac{c}{d}\right)^2 \\ \frac{bcs}{da}\in\mathbb{Z}}} a^{k-1} E_{k,\underline{L}_l,\frac{s}{c}}.$$

Note that bcs can be written as

$$bcs = b\left(s,\frac{c}{d}\right)^2 d\frac{sc/d}{(s,c/d)^2},$$

in other words the condition that $\frac{bcs}{da} \in \mathbb{Z}$ is superfluous. Thus, the right-hand side of (4.17) is equal to

$$\frac{1}{l^{k-1}} \prod_{p|l} \frac{1}{1+p^{-(k-1)}} \sum_{d|c} \mu(d) \sum_{s\in\mathbb{Z}_{(c)}} \sigma_{k-1}\left(b\left(s,\frac{c}{d}\right)^2\right) E_{k,\underline{L}_l,\frac{s}{c}}.$$

The term corresponding to $s = 0$ in the above equation is equal to

$$\frac{1}{l^{k-1}} \prod_{p|l} \frac{1}{1+p^{-(k-1)}} \sum_{d^2|l} \mu(d)\sigma_{k-1}\left(\frac{l}{d^2}\right) E_{k,\underline{L}_l,0}$$

and the above discussion implies that this expression is equal to $E_{k,\underline{L}_l,0}$. When $c = 1$ (i.e. when l is square-free), this term is the only one which arises. We claim that, for every

$c > 1$, every $s \in \mathbb{Z}_{(c)} \setminus 0$ and every square-free b, the following holds:

$$\sum_{d|c} \mu(d)\sigma_{k-1}\left(b\left(s, \frac{c}{d}\right)^2\right) = 0.$$

Let $c = \prod_{i=1}^{t} p_i^{a_i}$ be the prime decomposition of c and let $s = \prod_{i=1}^{t} p_i^{b_i} \prod_{j=t+1}^{r} q_j^{b_j}$ be that of s (with at least one $b_i > 0$ for some i in $\{1, \ldots, r\}$). Then

$$\sum_{d|c} \mu(d)\sigma_{k-1}\left(b\left(s, \frac{c}{d}\right)^2\right) = \sum_{c_1,\ldots,c_t=0}^{1} (-1)^{\#\{c_i\neq 0\}}\sigma_{k-1}\left(b\prod_{i=1}^{t} p_i^{\min\{2b_i, 2a_i-2c_i\}}\right).$$

Since $s < c$, there exists some some j in $\{1, \ldots, t\}$ such that $b_j < a_j$. For every such j, we have $\min\{2b_j, 2a_j - 2c_j\} = 2b_j$ and therefore

$$\begin{aligned}
&\sum_{c_1,\ldots,c_t=0}^{1} (-1)^{\#\{c_i\neq 0\}}\sigma_{k-1}\left(b\prod_{i=1}^{t} p_i^{\min\{2b_i, 2a_i-2c_i\}}\right) \\
&= \sum_{c_j=0}^{1} \sum_{\substack{c_1,\ldots,c_t=0\\ i\neq j}}^{1} (-1)^{\#\{c_i\neq 0: i\in\{1,\ldots,t\}\}}\sigma_{k-1}\left(bp_j^{2b_j}\prod_{\substack{i=1\\ i\neq j}}^{t} p_i^{\min\{2b_i, 2a_i-2c_i\}}\right) \\
&= \sum_{\substack{c_1,\ldots,c_t=0\\ i\neq j}}^{1} (-1)^{\#\{c_i\neq 0: i\neq j\}}\sigma_{k-1}\left(bp_j^{2b_j}\prod_{\substack{i=1\\ i\neq j}}^{t} p_i^{\min\{2b_i, 2a_i-2c_i\}}\right) \\
&\quad - \sum_{\substack{c_1,\ldots,c_t=0\\ i\neq j}}^{1} (-1)^{\#\{c_i\neq 0: i\neq j\}}\sigma_{k-1}\left(bp_j^{2b_j}\prod_{\substack{i=1\\ i\neq j}}^{t} p_j^{\min\{2b_i, 2a_i-2c_i\}}\right) = 0,
\end{aligned}$$

as claimed. Thus, we obtain that

$$\frac{1}{l^{k-1}}\prod_{p|l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2|l}\mu(d)V\left(\frac{l}{d^2}\right)\circ U(d)E_{k,\underline{L}_1,0} = E_{k,\underline{L}_l,0}$$

for every l in $\mathbb{N}$. □

When $\underline{L} = \underline{L}_m$ for some m in $\mathbb{N}$, the right-hand side of (4.17) is equal to

$$\frac{1}{l^{k-1}}\prod_{p|l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2|l}\mu(d)\sum_{\substack{x\in\mathbb{Z}_{(l)}\\ \frac{mx^2}{l}\in\mathbb{Z}}}\sum_{\substack{a|\left(\frac{mx^2}{l}, \frac{l}{d^2}\right)\\ \frac{x}{da}\in\mathbb{Z}}} a^{k-1}E_{k,\underline{L}_{ml},\frac{x}{l}}.$$

For every l in $\mathbb{N}$ which is coprime to m, calculations similar to those carried out in the proof of Lemma 4.44 imply that

$$\frac{1}{l^{k-1}}\prod_{p|l}\frac{1}{1+p^{-(k-1)}}\sum_{d^2|l}\mu(d)V\left(\frac{l}{d^2}\right)\circ U(d)E_{k,\underline{L}_m,0} = E_{k,\underline{L}_{ml},0}.$$

If $(l, m) > 1$, then the previous simplifications no longer hold.

APPENDIX A

Tables of Fourier coefficients

This chapter contains the tables used in Section 3.3. The code which generates them is available at `https://github.com/andreeamocanu/eigenvalues-Dn`. The difficulty of computing the Fourier coefficients (and implicitly the Hecke eigenvalues) decreases as the rank of the lattice increases, since the Fourier coefficients of Jacobi forms of index D_n ($n = 1, 3, 5$ and 7) are linear functions of representation numbers of quadratic forms in $8 - n$ variables. It also increases with the weight for fixed n.

We remind the reader that

$$D_n^{\#}/D_n = \left\{0, e_n, \frac{e_1 + \cdots + e_n}{2}, \frac{e_1 + \cdots + e_{n-1} - e_n}{2}\right\}$$

for every n in $\mathbb{N}$, where $\{e_i\}_i$ denotes the standard basis of $\mathbb{Z}^n$, and

$$D_{D_n} \simeq \left(\mathbb{Z}/4\mathbb{Z}, r \mapsto \frac{nr^2}{8} \bmod \mathbb{Z}\right)$$

when n is odd. Set $r_1^n := 0$, $r_2^n := e_n$, $r_3^n := \frac{e_1+\cdots+e_n}{2}$ and $r_4^n := \frac{e_1+\cdots+e_{n-1}-e_n}{2}$ for every n in $\{1, 3, 5, 7\}$. Then $-r_4^n = r_3^n$ in $D_n^{\#}/D_n$ and Proposition 1.25 implies that

$$C_\phi(D, r_4^n) = \begin{cases} -C_\phi(D, r_3^n), & \text{if } \phi \in J_{2k+1,D_n} \text{ and} \\ C_\phi(D, r_3^n), & \text{if } \phi \in J_{2k,D_n} \end{cases}$$

for every D in $\mathbb{Q}_{\leq 0}$ such that $(D, r_4^n) \in \mathrm{supp}(D_n)$. This can also be seen by inspecting the formulas for the Fourier coefficients of E_{4,D_n} and ψ_{12-n,D_n} obtained in Subsection 3.3.1. Furthermore, equation (3.21) implies that

$$C_\phi(D, r_1^n) = C_\phi(E, r_2^n) = 0$$

for every ϕ in J_{2k+1,D_n} and every (D, r_1^n) and (E, r_2^n) in $\mathrm{supp}(D_n)$.

In this chapter, we list the Fourier coefficients $C_\phi(D, r_j^n)$ of some of the Jacobi forms ϕ in J_{k,D_n}, for the first 100 values of D such that $(D, r_j^n) \in \mathrm{supp}(D_n)$ for every j, plus the Fourier coefficients which needed to compute their Hecke eigenvalues in Section 3.3.

Table A.1: Fourier coefficients of $11\psi_{8,D_1}$

D	$C_{11\psi_{8,D_1}}(D, r_1^1)$	D	$C_{11\psi_{8,D_1}}(D, r_2^1)$	D	$C_{11\psi_{8,D_1}}(D, r_3^1)$
-1	864	-1/2	144	-7/8	-1152
-2	-9216	-3/2	0	-15/8	10368
-3	36288	-5/2	-12096	-23/8	-24192
-4	-55296	-7/2	73728	-31/8	-48384
-5	4032	-9/2	-159408	-39/8	279936
-6	0	-11/2	0	-47/8	-145152
-7	114048	-13/2	536256	-55/8	-975744
-8	589824	-15/2	-663552	-63/8	1275264
-9	-2216160	-17/2	-24192	-71/8	1247616
-10	774144	-19/2	0	-79/8	-2158848
-11	3985344	-21/2	279936	-87/8	-653184

-12	-2322432	-23/2	1548288	-95/8	-1344384
-13	-2987712	-25/2	-1675440	-103/8	2911104
-14	-4718592	-27/2	0	-111/8	5878656
-15	11477376	-29/2	-4584384	-119/8	-3158784
-16	3538944	-31/2	3096576	-127/8	-2165760
-17	-15474816	-33/2	8781696	-135/8	-19035648
-18	10202112	-35/2	0	-143/8	3459456
-19	-2987712	-37/2	-5974848	-151/8	40606848
-20	-258048	-39/2	-17915904	-159/8	-10088064
-21	-9880704	-41/2	9144576	-167/8	-3459456
-22	0	-43/2	0	-175/8	-22596480
-23	25292160	-45/2	31026240	-183/8	-31632768
-24	0	-47/2	9289728	-191/8	31608576
-25	16947360	-49/2	-45472752	-199/8	-5056128
-26	-34320384	-51/2	0	-207/8	62052480
-27	-66624768	-53/2	-34098624	-215/8	-11200896
-28	-7299072	-55/2	62447616	-223/8	-41448960
-29	124655040	-57/2	-12037248	-231/8	81430272
-30	42467328	-59/2	0	-239/8	-113073408
-31	-48432384	-61/2	47863872	-247/8	-94753152
-32	-37748736	-63/2	-81616896	-255/8	31352832
-33	-43110144	-65/2	52026624	-263/8	107063424
-34	1548288	-67/2	0	-271/8	89687808
-35	-105303168	-69/2	-24240384	-279/8	53561088
-36	141834240	-71/2	-79847424	-287/8	46365696
-37	97516224	-73/2	22458240	-295/8	-191509632
-38	0	-75/2	0	-303/8	-167225472
-39	-83161728	-77/2	74739456	-311/8	-121129344
-40	-49545216	-79/2	138166272	-319/8	178003584
-41	54294912	-81/2	63090576	-327/8	-37231488
-42	-17915904	-83/2	0	-335/8	201930624
-43	103322304	-85/2	-501512832	-343/8	499313664
-44	-255062016	-87/2	41803776	-351/8	-513962496
-45	-4463424	-89/2	-5975424	-359/8	-15389568
-46	-99090432	-91/2	0	-367/8	23466240
-47	396184320	-93/2	519841152	-375/8	41368320
-48	148635648	-95/2	86040576	-383/8	-201712896
-49	-272836512	-97/2	-100920960	-391/8	-488017152
-50	107228160	-99/2	0	-399/8	350479872
-51	88739712	-101/2	-287872704	-407/8	-197797248
-52	191213568	-103/2	-186310656	-415/8	616321152
-53	-932001984	-105/2	-312139008	-423/8	372314880
-54	0	-107/2	0	-431/8	20659968
-55	76374144	-109/2	988835904	-439/8	-164336256
-56	301989888	-111/2	-376233984	-447/8	53198208
-57	194856192	-113/2	146375424	-455/8	309768192
-58	293400576	-115/2	0	-463/8	-1680346368
-59	734287680	-117/2	-593635392	-471/8	-497446272
-60	-734552064	-119/2	202162176	-479/8	78769152

-61	-245802816	-121/2	488351952	-487/8	709031808
-62	-198180864	-123/2	0	-495/8	1080148608
-63	-292533120	-125/2	-48263040	-503/8	502879104
-64	-226492416	-127/2	138608640	-511/8	87717888
-65	308297088	-129/2	-781861248	-519/8	114058368
-66	-562028544	-131/2	0	-527/8	-130540032
-67	215383104	-133/2	-1234737792	-535/8	-1230784128
-68	990388224	-135/2	1218281472	-543/8	38351232
-69	1187488512	-137/2	381867264	-551/8	-789360768
-70	0	-139/2	0	-559/8	256330368
-71	-1358822016	-141/2	1255004928	-567/8	-504724608
-72	-652935168	-143/2	-221405184	-575/8	-474526080
-73	145926144	-145/2	566592768	-583/8	1126705536
-74	382390272	-147/2	0	-591/8	876935808
-75	-422210880	-149/2	-1590591168	-599/8	845500032
-76	191213568	-151/2	-2598838272	-607/8	426085632
-77	-710620416	-153/2	62052480	-615/8	-1461825792
-78	1146617856	-155/2	0	-623/8	-233805312
-79	-1261018368	-157/2	2138052672	-631/8	4746234240
-80	16515072	-159/2	645636096	-639/8	-3200135040
-81	2691374688	-161/2	1945631232	-647/8	-1762701696
-82	-585252864	-163/2	0	-655/8	-2152621440
-83	-1886472000	-165/2	-745189632	-663/8	-1615396608
-84	632365056	-167/2	221405184	-671/8	2307457152
-85	1911192192	-169/2	-2261559600	-679/8	-2459255040
-86	0	-171/2	0	-687/8	2897057664
-87	-1212962688	-173/2	-227199168	-695/8	507178368
-88	0	-175/2	1446174720	-703/8	3809957760
-89	-1369734912	-177/2	-3320341632	-711/8	2389844736
-90	-1985679360	-179/2	0	-719/8	724211712
-91	1515411072	-181/2	1939105728	-727/8	-3018395520
-92	-1618698240	-183/2	2024497152	-735/8	-3274038144
-93	156981888	-185/2	-104702976	-743/8	1180158336
-94	-594542592	-187/2	0	-751/8	-3332448000
-95	7939228032	-189/2	-513962496	-759/8	474211584
-96	0	-191/2	-2022948864	-767/8	-3678210432
-97	-1462962816	-193/2	1335005568	-775/8	562947840
-98	2910256128	-195/2	0	-783/8	1199245824
-99	-4411775808	-197/2	1156344768	-791/8	-204360192
-100	-1084631040	-199/2	323592192	-799/8	2969459712
-121	-131145696				
-169	-5228631648				
-225	-43469978400				
-289	31477013856				
-361	-250713464544				
-441	699825653280				
-529	-395764329888				
-625	-987067533600				

Table A.2: Fourier coefficients of ψ_{8,D_3}

D	$C_{\psi_{8,D_3}}(D,r_1^3)$	D	$C_{\psi_{8,D_3}}(D,r_2^3)$	D	$C_{\psi_{8,D_3}}(D,r_3^3)$
-1	2	-1/2	1	-5/8	-2
-2	-24	-3/2	-12	-13/8	22
-3	108	-5/2	56	-21/8	-84
-4	-176	-7/2	-112	-29/8	66
-5	-196	-9/2	9	-37/8	398
-6	1056	-11/2	364	-45/8	-990
-7	-728	-13/2	-616	-53/8	-70
-8	-1472	-15/2	432	-61/8	2354
-9	990	-17/2	-240	-69/8	-1080
-10	2752	-19/2	-484	-77/8	-1848
-11	1276	-21/2	2352	-85/8	-2292
-12	-9504	-23/2	-2608	-93/8	3852
-13	772	-25/2	1705	-101/8	7682
-14	9856	-27/2	-3024	-109/8	-8430
-15	1032	-29/2	-1848	-117/8	198
-16	128	-31/2	11168	-125/8	-9660
-17	-13576	-33/2	-6480	-133/8	-5012
-18	-216	-35/2	5432	-141/8	33048
-19	-2620	-37/2	-11144	-149/8	994
-20	17248	-39/2	-12720	-157/8	-6158
-21	22680	-41/2	16320	-165/8	-36984
-22	-32032	-43/2	5964	-173/8	-9126
-23	-9176	-45/2	27720	-181/8	39558
-24	-768	-47/2	-15904	-189/8	-21168
-25	15910	-49/2	-33551	-197/8	45206
-26	-30272	-51/2	-20520	-205/8	22616
-27	27216	-53/2	1960	-213/8	-36504
-28	64064	-55/2	26608	-221/8	-44224
-29	-51348	-57/2	65520	-229/8	-19626
-30	-38016	-59/2	18244	-237/8	-11004
-31	-31920	-61/2	-65912	-245/8	-126
-32	84480	-63/2	-1008	-253/8	121596
-33	-29568	-65/2	-111360	-261/8	32670
-34	5760	-67/2	33468	-269/8	15202
-35	20440	-69/2	30240	-277/8	-138758
-36	-87120	-71/2	45392	-285/8	38016
-37	71436	-73/2	145200	-293/8	-40018
-38	42592	-75/2	-95460	-301/8	-292376
-39	41256	-77/2	51744	-309/8	258852
-40	-180736	-79/2	-79968	-317/8	112558
-41	51720	-81/2	-174879	-325/8	175010
-42	115584	-83/2	-43404	-333/8	3582
-43	-100852	-85/2	64176	-341/8	-167836
-44	-112288	-87/2	26352	-349/8	-53126
-45	-1764	-89/2	267600	-357/8	-182952
-46	229504	-91/2	122584	-365/8	145772

-47	48976	-93/2	-107856	-373/8	-75082
-48	6912	-95/2	-41936	-381/8	-183876
-49	-67102	-97/2	-357360	-389/8	-23974
-50	-40920	-99/2	180180	-397/8	531186
-51	-133512	-101/2	-215096	-405/8	104814
-52	-67936	-103/2	-66288	-413/8	-2996
-53	-201612	-105/2	272160	-421/8	95318
-54	266112	-107/2	23300	-429/8	-323136
-55	215864	-109/2	236040	-437/8	-134596
-56	-7168	-111/2	342096	-445/8	-406232
-57	56160	-113/2	-188640	-453/8	-160860
-58	-90816	-115/2	-490760	-461/8	107514
-59	278324	-117/2	-5544	-469/8	536928
-60	-90816	-119/2	-354592	-477/8	-34650
-61	126588	-121/2	373561	-485/8	198236
-62	-982784	-123/2	-75816	-493/8	253876
-63	-360360	-125/2	270480	-501/8	470880
-64	357376	-127/2	29440	-509/8	-160070
-65	596840	-129/2	-422640	-517/8	-866668
-66	155520	-131/2	473196	-525/8	-143220
-67	-205100	-133/2	140336	-533/8	-751788
-68	1194688	-135/2	108864	-541/8	405790
-69	-362544	-137/2	-46080	-549/8	21186
-70	-478016	-139/2	-157980	-557/8	395014
-71	-202776	-141/2	-925344	-565/8	-786124
-72	-13248	-143/2	217360	-573/8	1355184
-73	-727696	-145/2	428160	-581/8	1574580
-74	-547648	-147/2	-756	-589/8	-1554160
-75	184140	-149/2	-27832	-597/8	245700
-76	230560	-151/2	291696	-605/8	-747122
-77	344848	-153/2	-118800	-613/8	573526
-78	1119360	-155/2	110320	-621/8	-272160
-79	592016	-157/2	172424	-629/8	-508820
-80	-12544	-159/2	-565488	-637/8	1386
-81	-104814	-161/2	-134400	-645/8	-1300320
-82	-391680	-163/2	223516	-653/8	-1053366
-83	-290756	-165/2	1035552	-661/8	1394494
-84	-1995840	-167/2	-382928	-669/8	1304484
-85	567496	-169/2	-949031	-677/8	1062802
-86	-524832	-171/2	-4356	-685/8	1917412
-87	685176	-173/2	255528	-693/8	-914760
-88	23296	-175/2	-890960	-701/8	-921658
-89	-301376	-177/2	2093040	-709/8	-682394
-90	1362240	-179/2	-750228	-717/8	-872208
-91	342888	-181/2	-1107624	-725/8	112530
-92	807488	-183/2	678960	-733/8	-131238
-93	309960	-185/2	-775200	-741/8	-2876328
-94	1399552	-187/2	714296	-749/8	1793008
-95	-2209528	-189/2	592704	-757/8	64838

-96	-2144256	-191/2	1157920	-765/8	-20628
-97	-399848	-193/2	-119280	-773/8	931542
-98	805224	-195/2	-379728	-781/8	618572
-99	11484	-197/2	-1265768	-789/8	2421360
-100	-1400080	-199/2	1603536	-797/8	-1044646
-121	1391326				
-169	-412890				
-225	7875450				
-289	-16651582				
-361	26275038				
-441	-33215490				
-529	24413858				
-625	-20810950				
-729	-201788658				
-841	297835558				
-961	-162944638				
-1089	688706370				
-1225	-533796410				
-1369	-225738714				
-1521	-204380550				
-1681	384528482				
-1849	259765470				
-2025	-833795370				
-2209	4916006978				
-2401	-2831097598				

Table A.3: Fourier coefficients of ψ_{9,D_3} and ψ_{7,D_5}

D	$C_{\psi_{9,D_3}}(D,r_3^3)$	D	$C_{\psi_{7,D_5}}(D,r_5^3)$
-5/8	1	-3/8	1
-13/8	-15	-11/8	-9
-21/8	90	-19/8	27
-29/8	-245	-27/8	-12
-37/8	105	-35/8	-90
-45/8	1107	-43/8	135
-53/8	-2485	-51/8	54
-61/8	195	-59/8	-99
-69/8	4860	-67/8	-189
-77/8	-2420	-75/8	-85
-85/8	-3990	-83/8	657
-93/8	-8190	-91/8	-162
-101/8	19695	-99/8	-135
-109/8	13755	-107/8	-171
-117/8	-38475	-115/8	-810
-125/8	3990	-123/8	702
-133/8	-9750	-131/8	495
-141/8	34020	-139/8	837
-149/8	43015	-147/8	-673
-157/8	-46605	-155/8	-900
-165/8	-13860	-163/8	243
-173/8	-127385	-171/8	-1053
-181/8	106485	-179/8	-297
-189/8	165240	-187/8	1566
-197/8	-79275	-195/8	2700
-205/8	-16380	-203/8	-1764
-213/8	-92340	-211/8	81
-221/8	-35840	-219/8	-1188
-229/8	-151995	-227/8	-1377
-237/8	188550	-235/8	270
-245/8	315783	-243/8	-2043
-253/8	90090	-251/8	3321
-261/8	-271215	-259/8	-756
-269/8	-307485	-267/8	3726
-277/8	20475	-275/8	3015
-285/8	-505440	-283/8	-4563
-293/8	915385	-291/8	-3348
-301/8	209340	-299/8	504
-309/8	-284130	-307/8	-351
-317/8	337645	-315/8	-1350
-325/8	-294225	-323/8	-468
-333/8	269325	-331/8	-891
-341/8	-1707970	-339/8	7074
-349/8	-70305	-347/8	1611
-357/8	1297620	-355/8	2700
-365/8	574210	-363/8	-2423

-373/8	492765	-371/8	-1512
-381/8	251370	-379/8	-3267
-389/8	-847245	-387/8	-5265
-397/8	-1102725	-395/8	-1800
-405/8	438129	-403/8	3510
-413/8	-1416190	-411/8	2970
-421/8	641445	-419/8	-6741
-429/8	0	-427/8	8910
-437/8	1537330	-435/8	-1620
-445/8	1239420	-443/8	7227
-453/8	-800370	-451/8	7506
-461/8	1403815	-459/8	-648
-469/8	-472080	-467/8	-13923
-477/8	-2750895	-475/8	-9045
-485/8	-2707950	-483/8	7884
-493/8	-761490	-491/8	-5985
-501/8	952560	-499/8	-2079
-509/8	7162255	-507/8	-815
-517/8	-1869450	-515/8	6930
-525/8	-1047150	-523/8	1107
-533/8	1169350	-531/8	-1485
-541/8	728805	-539/8	12231
-549/8	500175	-547/8	5049
-557/8	-5564055	-555/8	-8100
-565/8	1575990	-563/8	15075
-573/8	-5375160	-571/8	-11205
-581/8	939590	-579/8	-4104
-589/8	1736280	-587/8	-10719
-597/8	4629870	-595/8	-18900
-605/8	151789	-603/8	7371
-613/8	2775045	-611/8	5310
-621/8	8922960	-619/8	7587
-629/8	-9469990	-627/8	-756
-637/8	-4736745	-635/8	6390
-645/8	-408240	-643/8	-11799
-653/8	-2820545	-651/8	16632
-661/8	-7843095	-659/8	4041
-669/8	4410630	-667/8	13500
-677/8	7312455	-675/8	1020
-685/8	4813590	-683/8	-9117
-693/8	-2678940	-691/8	-297
-701/8	-1306095	-699/8	-23274
-709/8	4328205	-707/8	-12078
-717/8	4383720	-715/8	9180
-725/8	2850575	-723/8	-4212
-733/8	650415	-731/8	-16236
-741/8	-16312140	-739/8	5535
-749/8	-13250360	-747/8	9855
-757/8	3613365	-755/8	29700

-765/8	-10234350	-763/8	-3780
-773/8	25950965	-771/8	-2862
-781/8	-2986830	-779/8	-11034
-789/8	-126360	-787/8	23463
-797/8	14627375	-795/8	13500
-845/8	-6051657	-891/8	21303
-2925/8	-754687125	-867/8	19619
-1445/8	84706867	-1083/8	46799
-1805/8	290177621	-539/8	12231
-2205/8	349571781	-4851/8	183465
-2645/8	458060567	-1587/8	-80879
-8125/8	17136589125	-1875/8	-60275
-3645/8	-960510717	-8019/8	39609
-4205/8	4663815989	-2523/8	-78181
-4805/8	2711816609	-2883/8	-197761
-5445/8	168030423	-1331/8	9828
-15925/8	-92911253175	-11979/8	147420
-6845/8	-440148993	-3675/8	57205
-7605/8	-6699184299	-4107/8	109873
-8405/8	-54454984999	-1859/8	-32211
-9245/8	-65088056133	-16731/8	-483165
-1053/8	-46725255	-5043/8	79763
-26325/8	-916515876825	-5547/8	710255
-11045/8	31316662703	-22275/8	-7136505
-12005/8	39981446849	-6627/8	-576479
		-7203/8	-139775
		-3179/8	-176571
		-28611/8	-2648565
		-8427/8	-1345141
		-9075/8	205955
		-3971/8	-297729
		-35739/8	-4465935
		-10443/8	-2846039
		-11163/8	600721
		-43659/8	-28950777
		-12675/8	69275
		-13467/8	426767
		-5819/8	508905
		-52371/8	7633575
		-15123/8	1056817
		-15987/8	591265
		-6875/8	69975
		-61875/8	1049625
		-17787/8	1630679
		-18723/8	4059839
		-72171/8	-47064969
		-2299/8	6453
		-20691/8	-251667

Table A.4: Fourier coefficients of $7\psi_{8,D_5}$

D	$C_{7\psi_{8,D_5}}(D,r_1^5)$	D	$C_{7\psi_{8,D_5}}(D,r_2^5)$	D	$C_{7\psi_{8,D_5}}(D,r_3^5)$
-1	96	-1/2	-48	-3/8	24
-2	-768	-3/2	384	-11/8	-312
-3	1728	-5/2	-960	-19/8	1608
-4	1536	-7/2	0	-27/8	-3744
-5	-10560	-9/2	3600	-35/8	1680
-6	6144	-11/2	-4992	-43/8	9672
-7	18816	-13/2	2112	-51/8	-15984
-8	-12288	-15/2	0	-59/8	-936
-9	-22752	-17/2	-11136	-67/8	3912
-10	-15360	-19/2	25728	-75/8	35880
-11	53952	-21/2	-8064	-83/8	-11304
-12	27648	-23/2	0	-91/8	-100464
-13	-40896	-25/2	-11760	-99/8	73944
-14	0	-27/2	-59904	-107/8	40536
-15	-97920	-29/2	80064	-115/8	40080
-16	24576	-31/2	0	-123/8	-57456
-17	126336	-33/2	65664	-131/8	-111000
-18	57600	-35/2	26880	-139/8	56664
-19	-68544	-37/2	-228288	-147/8	-80472
-20	-168960	-39/2	0	-155/8	233760
-21	169344	-41/2	-31488	-163/8	84552
-22	-79872	-43/2	154752	-171/8	-120600
-23	-104064	-45/2	227520	-179/8	-203160
-24	98304	-47/2	0	-187/8	-15216
-25	143520	-49/2	-69552	-195/8	-73440
-26	33792	-51/2	-255744	-203/8	-61152
-27	-269568	-53/2	-224064	-211/8	661464
-28	301056	-55/2	0	-219/8	117792
-29	-222528	-57/2	157824	-227/8	-53400
-30	0	-59/2	-14976	-235/8	-1054320
-31	185088	-61/2	725184	-243/8	111672
-32	-196608	-63/2	0	-251/8	458616
-33	-223488	-65/2	-234240	-259/8	-362208
-34	-178176	-67/2	62592	-267/8	668304
-35	900480	-69/2	-1029888	-275/8	-76440
-36	-364032	-71/2	0	-283/8	305496
-37	-206400	-73/2	-180096	-291/8	48672
-38	411648	-75/2	574080	-299/8	-209856
-39	584064	-77/2	715008	-307/8	-852264
-40	-245760	-79/2	0	-315/8	-398160
-41	-1320576	-81/2	383184	-323/8	-280032
-42	-129024	-83/2	-180864	-331/8	1197144
-43	-436800	-85/2	-769920	-339/8	263088
-44	863232	-87/2	0	-347/8	44520
-45	792000	-89/2	436608	-355/8	1577760
-46	0	-91/2	-1607424	-363/8	-996168

-47	160512	-93/2	434304	-371/8	-348096
-48	442368	-95/2	0	-379/8	-1567656
-49	139104	-97/2	603264	-387/8	-725400
-50	-188160	-99/2	1183104	-395/8	352320
-51	-1120896	-101/2	232896	-403/8	1090704
-52	-654336	-103/2	0	-411/8	-1292112
-53	-363456	-105/2	-1693440	-419/8	2329992
-54	-958464	-107/2	648576	-427/8	2254224
-55	167040	-109/2	172224	-435/8	-303840
-56	0	-111/2	0	-443/8	-1280280
-57	3739392	-113/2	-1496832	-451/8	-3861072
-58	1281024	-115/2	641280	-459/8	2493504
-59	-391104	-117/2	-158400	-467/8	-1604616
-60	-1566720	-119/2	0	-475/8	393960
-61	945600	-121/2	1992336	-483/8	1336608
-62	0	-123/2	-919296	-491/8	-1972728
-63	-4459392	-125/2	-835200	-499/8	-1481256
-64	393216	-127/2	0	-507/8	3588792
-65	-1633920	-129/2	1081728	-515/8	3115440
-66	1050624	-131/2	-1776000	-523/8	198120
-67	227136	-133/2	2400384	-531/8	221832
-68	2021376	-135/2	0	-539/8	-452088
-69	799488	-137/2	-344832	-547/8	-4342824
-70	430080	-139/2	906624	-555/8	-4220640
-71	3471744	-141/2	-1859328	-563/8	1452168
-72	921600	-143/2	0	-571/8	1108200
-73	-1629696	-145/2	-1770240	-579/8	615744
-74	-3652608	-147/2	-1287552	-587/8	-1979016
-75	423360	-149/2	1154496	-595/8	6185760
-76	-1096704	-151/2	0	-603/8	-293400
-77	-2972928	-153/2	2639232	-611/8	3118800
-78	0	-155/2	3740160	-619/8	13800
-79	1475328	-157/2	-6352704	-627/8	-1255392
-80	-2703360	-159/2	0	-635/8	-1431600
-81	1659744	-161/2	2333184	-643/8	-2338344
-82	-503808	-163/2	1352832	-651/8	-3358656
-83	8149440	-165/2	5679360	-659/8	-4064232
-84	2709504	-167/2	0	-667/8	6075168
-85	-3288960	-169/2	-7177584	-675/8	-5597280
-86	2476032	-171/2	-1929600	-683/8	1922280
-87	-3998592	-173/2	-3057216	-691/8	5664360
-88	-1277952	-175/2	0	-699/8	3963600
-89	-4253952	-177/2	-763776	-707/8	-91728
-90	3640320	-179/2	-3250560	-715/8	-1878240
-91	-2072448	-181/2	6600000	-723/8	-587232
-92	-1665024	-183/2	0	-731/8	5560416
-93	-2477952	-185/2	6213120	-739/8	-4968600
-94	0	-187/2	-243456	-747/8	2679048
-95	4817280	-189/2	1257984	-755/8	-6210720

-96	1572864	-191/2	0	-763/8	-9845472
-97	3273600	-193/2	2077824	-771/8	8972208
-98	-1112832	-195/2	-1175040	-779/8	-1915440
-99	-4046400	-197/2	-926400	-787/8	-3989400
-100	2296320	-199/2	0	-795/8	-4105440
-121	-6795744				
-169	19838880				
-225	-34014240				
-289	-31793568				
-361	17765664				
-441	-32967648				
-529	46497120				
-625	-62637600				
-729	188907552				
-841	-300784224				
-961	-37045152				
-1089	1610591328				
-1225	207960480				
-1369	593398944				
-1521	-4701814560				
-1681	-126117024				
-1849	2904173088				
-2025	2481317280				
-2209	-547333536				
-2401	-4006373280				
-2601	7535075616				
-2809	2352642720				
-3025	-10159637280				
-3249	-4210462368				
-3481	9611302944				
-3721	-1857732192				
-3969	2404969056				
-4225	29659125600				
-4489	-13586243808				
-4761	-11019817440				
-5041	-34729060512				
-5329	21776815968				
-5625	14845111200				
-5929	-9847033056				
-6241	50980984416				
-6561	-62138319264				

Table A.5: Fourier coefficients of ψ_{10,D_5}

D	$C_{\psi_{10,D_5}}(D, r_1^5)$	D	$C_{\psi_{10,D_5}}(D, r_2^5)$	D	$C_{\psi_{10,D_5}}(D, r_3^5)$
-1	-78	-1/2	3	-3/8	3
-2	192	-3/2	192	-11/8	33
-3	1332	-5/2	-1020	-19/8	-663
-4	-4992	-7/2	0	-27/8	3708
-5	660	-9/2	5895	-35/8	-9870
-6	12288	-11/2	2112	-43/8	13809
-7	-7224	-13/2	-32604	-51/8	-13374
-8	12288	-15/2	0	-59/8	15651
-9	-39546	-17/2	94728	-67/8	15321
-10	-65280	-19/2	-42432	-75/8	-125475
-11	152724	-21/2	-44856	-83/8	156267
-12	85248	-23/2	0	-91/8	20706
-13	-157284	-25/2	-219225	-99/8	16731
-14	0	-27/2	237312	-107/8	-344637
-15	-50040	-29/2	326124	-115/8	10770
-16	-319488	-31/2	0	-123/8	766818
-17	194856	-33/2	-323928	-131/8	-525795
-18	377280	-35/2	-631680	-139/8	-160749
-19	713628	-37/2	440004	-147/8	-160251
-20	42240	-39/2	0	-155/8	1215780
-21	-1780632	-41/2	122928	-163/8	-1203351
-22	135168	-43/2	883776	-171/8	-1302795
-23	-1043256	-45/2	-517140	-179/8	1867485
-24	786432	-47/2	0	-187/8	1244562
-25	3262350	-49/2	545643	-195/8	955620
-26	-2086656	-51/2	-855936	-203/8	-4422684
-27	1646352	-53/2	-2740212	-211/8	390699
-28	-462336	-55/2	0	-219/8	-574092
-29	-881484	-57/2	3887208	-227/8	2180685
-30	0	-59/2	1001664	-235/8	2864130
-31	-6705264	-61/2	-424404	-243/8	-199881
-32	786432	-63/2	0	-251/8	-2116569
-33	2878128	-65/2	-2522640	-259/8	-3726492
-34	6062592	-67/2	980544	-267/8	3007602
-35	5124840	-69/2	1384272	-275/8	-2411475
-36	-2530944	-71/2	0	-283/8	4629267
-37	-3584220	-73/2	-6748248	-291/8	-4366188
-38	-2715648	-75/2	-8030400	-299/8	7027176
-39	-4924728	-77/2	14972496	-307/8	-8021637
-40	-4177920	-79/2	0	-315/8	-5004090
-41	6841608	-81/2	2503251	-323/8	15365076
-42	-2870784	-83/2	10001088	-331/8	8306019
-43	-76380	-85/2	-7623480	-339/8	-2594538
-44	9774336	-87/2	0	-347/8	-31328475
-45	1296900	-89/2	-14031432	-355/8	-5996220
-46	0	-91/2	1325184	-363/8	12708399

-47	6256272	-93/2	-8795448	-371/8	22356264
-48	5455872	-95/2	0	-379/8	-4431069
-49	-14186718	-97/2	13131528	-387/8	27134685
-50	-14030400	-99/2	1070784	-395/8	-17963160
-51	439848	-101/2	706524	-403/8	-26598702
-52	-10066176	-103/2	0	-411/8	-7738362
-53	-9775044	-105/2	34287120	-419/8	-7097247
-54	15187968	-107/2	-22056768	-427/8	31815042
-55	39366360	-109/2	3733164	-435/8	-6552540
-56	0	-111/2	0	-443/8	10931925
-57	-16798608	-113/2	-15265776	-451/8	-2653002
-58	20871936	-115/2	689280	-459/8	-16530264
-59	-39697092	-117/2	-64066860	-467/8	65247
-60	-3202560	-119/2	0	-475/8	48448725
-61	33062340	-121/2	12708399	-483/8	-59365404
-62	0	-123/2	49076352	-491/8	20729817
-63	-3662568	-125/2	58599000	-499/8	9517131
-64	-20447232	-127/2	0	-507/8	9617871
-65	-30122040	-129/2	-13164552	-515/8	3819510
-66	-20731392	-131/2	-33650880	-523/8	-72738075
-67	11934636	-133/2	25319112	-531/8	7935057
-68	12470784	-135/2	0	-539/8	6002073
-69	69448464	-137/2	-68494224	-547/8	16284843
-70	-40427520	-139/2	-10287936	-555/8	2458980
-71	61783368	-141/2	18745488	-563/8	79803441
-72	24145920	-143/2	0	-571/8	16519605
-73	-58510464	-145/2	-37464720	-579/8	63567576
-74	28160256	-147/2	-10256064	-587/8	-166216953
-75	-97335900	-149/2	43792476	-595/8	-34758780
-76	45672192	-151/2	0	-603/8	30105765
-77	173712	-153/2	48027096	-611/8	-112658790
-78	0	-155/2	77809920	-619/8	-13830555
-79	-104534256	-157/2	-7972308	-627/8	117374004
-80	2703360	-159/2	0	-635/8	142240650
-81	75478338	-161/2	44085216	-643/8	73145547
-82	7867392	-163/2	-77014464	-651/8	-55777176
-83	61662660	-165/2	-125112240	-659/8	-94080813
-84	-113960448	-167/2	0	-667/8	-88511676
-85	53505720	-169/2	9617871	-675/8	-155087100
-86	56561664	-171/2	-83378880	-683/8	256087605
-87	81274968	-173/2	-16221828	-691/8	-24603555
-88	8650752	-175/2	0	-699/8	41126490
-89	36913584	-177/2	153272808	-707/8	27531126
-90	-33096960	-179/2	119519040	-715/8	-27172860
-91	-86768136	-181/2	-16462380	-723/8	-165738924
-92	-66768384	-183/2	0	-731/8	-33222564
-93	-157222728	-185/2	-81021120	-739/8	71095245
-94	0	-187/2	79651968	-747/8	79227369
-95	-99129720	-189/2	-55442016	-755/8	-35880660

-96	50331648	-191/2	0	-763/8	-66468444
-97	61951560	-193/2	-25078008	-771/8	140530518
-98	34921152	-195/2	61159680	-779/8	71155050
-99	300102660	-197/2	-43153260	-787/8	223092285
-100	208790400	-199/2	0	-795/8	-142360380
-121	-54054858				
-169	-1003046850				
-225	1654011450				
-289	-3664036974				
-361	-6964711962				
-441	-7192666026				
-529	81005125890				
-625	-92207163750				
-729	156340323126				
-841	23069261982				
-961	-427678395534				
-1089	-27405813006				
-1225	593359480350				
-1369	1327507544142				
-1521	-508544752950				
-1681	582993418722				
-1849	-3544394729226				
-2025	-3156881486850				
-2209	3337902191442				
-2401	6646101541170				
-2601	-1857666745818				
-2809	-8174228488530				
-3025	2260844435850				
-3249	-3531108964734				
-3481	34042857156438				
-3721	-56361407427234				
-3969	13728075593778				
-4225	41952434501250				
-4489	-23220449276154				
-4761	41069598826230				
-5041	20332412880546				
-5329	-108391819515486				
-5625	-46749032021250				
-5929	-9831551627898				
-6241	238946924093202				
-6561	72899789108562				

Table A.6: Fourier coefficients of $5\psi_{8,D_7}$

D	$C_{5\psi_{8,D_7}}(D,r_1^7)$	D	$C_{5\psi_{8,D_7}}(D,r_2^7)$	D	$C_{5\psi_{8,D_7}}(D,r_3^7)$
-1	864	-1/2	-144	-1/8	18
-2	1152	-3/2	-1728	-9/8	-270
-3	-1728	-5/2	0	-17/8	1728
-4	-6912	-7/2	6912	-25/8	-6030
-5	-8640	-9/2	2160	-33/8	12096
-6	13824	-11/2	1728	-41/8	-13824
-7	17280	-13/2	0	-49/8	12114
-8	-9216	-15/2	-34560	-57/8	-22464
-9	33696	-17/2	-13824	-65/8	34560
-10	0	-19/2	22464	-73/8	-1728
-11	-63936	-21/2	0	-81/8	-42606
-12	13824	-23/2	34560	-89/8	1728
-13	-60480	-25/2	48240	-97/8	29376
-14	-55296	-27/2	-20736	-105/8	86400
-15	86400	-29/2	0	-113/8	-134784
-16	55296	-31/2	69120	-121/8	-4302
-17	38016	-33/2	-96768	-129/8	8640
-18	-17280	-35/2	-120960	-137/8	145152
-19	50112	-37/2	0	-145/8	-172800
-20	69120	-39/2	-131328	-153/8	67392
-21	24192	-41/2	110592	-161/8	96768
-22	-13824	-43/2	98496	-169/8	-14670
-23	-203904	-45/2	0	-177/8	-243648
-24	-110592	-47/2	96768	-185/8	-120960
-25	-73440	-49/2	-96912	-193/8	416448
-26	0	-51/2	335232	-201/8	133056
-27	-20736	-53/2	0	-209/8	-105408
-28	-138240	-55/2	-172800	-217/8	-314496
-29	191808	-57/2	179712	-225/8	90450
-30	276480	-59/2	-216000	-233/8	-177984
-31	145152	-61/2	0	-241/8	247104
-32	73728	-63/2	-103680	-249/8	-150336
-33	-317952	-65/2	-276480	-257/8	518400
-34	110592	-67/2	-461376	-265/8	-172800
-35	604800	-69/2	0	-273/8	-231552
-36	-269568	-71/2	532224	-281/8	-181440
-37	191808	-73/2	13824	-289/8	176274
-38	-179712	-75/2	146880	-297/8	145152
-39	-756864	-77/2	0	-305/8	-86400
-40	0	-79/2	-41472	-313/8	-345600
-41	-134784	-81/2	340848	-321/8	-57024
-42	0	-83/2	191808	-329/8	411264
-43	-665280	-85/2	0	-337/8	196992
-44	511488	-87/2	269568	-345/8	950400
-45	129600	-89/2	-13824	-353/8	-787968
-46	-276480	-91/2	169344	-361/8	-842382

-47	698112	-93/2	0	-369/8	-539136
-48	-110592	-95/2	-864000	-377/8	286848
-49	581472	-97/2	-235008	-385/8	120960
-50	-385920	-99/2	67392	-393/8	1211328
-51	701568	-101/2	0	-401/8	98496
-52	483840	-103/2	-601344	-409/8	667008
-53	-772416	-105/2	-691200	-417/8	-1933632
-54	165888	-107/2	-22464	-425/8	-146880
-55	190080	-109/2	0	-433/8	945216
-56	442368	-111/2	366336	-441/8	-181710
-57	82944	-113/2	1078272	-449/8	-274752
-58	0	-115/2	293760	-457/8	-207360
-59	-1325376	-117/2	0	-465/8	-138240
-60	-691200	-119/2	96768	-473/8	616896
-61	-741312	-121/2	34416	-481/8	-127872
-62	-552960	-123/2	-604800	-489/8	-209088
-63	673920	-125/2	0	-497/8	1316736
-64	-442368	-127/2	2211840	-505/8	-518400
-65	432000	-129/2	-69120	-513/8	-269568
-66	774144	-131/2	98496	-521/8	-694656
-67	-661824	-133/2	0	-529/8	1017810
-68	-304128	-135/2	-414720	-537/8	589248
-69	919296	-137/2	-1161216	-545/8	-1641600
-70	967680	-139/2	-368064	-553/8	-2865024
-71	-176256	-141/2	0	-561/8	2536704
-72	138240	-143/2	-988416	-569/8	-473472
-73	1029888	-145/2	1382400	-577/8	1676160
-74	0	-147/2	-2348352	-585/8	1347840
-75	578880	-149/2	0	-593/8	1268352
-76	-400896	-151/2	-1223424	-601/8	-485568
-77	-546048	-153/2	-539136	-609/8	-2128896
-78	1050624	-155/2	2246400	-617/8	-2196288
-79	1112832	-157/2	0	-625/8	-139950
-80	-552960	-159/2	-103680	-633/8	-907200
-81	-1485216	-161/2	-774144	-641/8	2279232
-82	-884736	-163/2	2787264	-649/8	696384
-83	309312	-165/2	0	-657/8	25920
-84	-193536	-167/2	628992	-665/8	1779840
-85	-1641600	-169/2	117360	-673/8	-1266624
-86	-787968	-171/2	-336960	-681/8	-872640
-87	-1717632	-173/2	0	-689/8	-241920
-88	110592	-175/2	-587520	-697/8	383616
-89	-400896	-177/2	1949184	-705/8	-604800
-90	0	-179/2	98496	-713/8	1842048
-91	943488	-181/2	0	-721/8	774144
-92	1631232	-183/2	1845504	-729/8	79218
-93	3003264	-185/2	967680	-737/8	-3112128
-94	-774144	-187/2	-2056320	-745/8	259200
-95	-2678400	-189/2	0	-753/8	-1097280

-96	884736	-191/2	-2419200	-761/8	-1534464
-97	162432	-193/2	-3331584	-769/8	1162944
-98	775296	-195/2	172800	-777/8	3580416
-99	959040	-197/2	0	-785/8	-3162240
-100	587520	-199/2	1140480	-793/8	4223232
-121	2093472				
-169	3092256				
-225	-2864160				
-289	8461152				
-361	-28581984				
-441	22677408				
-529	48854880				
-625	-52077600				
-729	-91515744				
-841	-67548384				
-961	170865504				
-1089	81645408				
-1225	-49425120				
-1369	182458656				
-1521	120597984				
-1681	-50180256				
-1849	-476272224				
-2025	126243360				
-2209	318671712				
-2401	-120765600				
-2601	329984928				
-2809	-1162201824				
-3025	-177945120				
-3249	-1114697376				
-3481	2458977696				
-3721	911246112				
-3969	-999550368				
-4225	-262841760				
-4489	150991776				
-4761	1905340320				
-5041	-1531560096				
-5329	510852960				
-5625	-2031026400				
-5929	1408906656				
-6241	-3507700896				
-6561	2149978464				

Table A.7: Fourier coefficients of $4\psi_{10,D_7}$

D	$C_{4\psi_{10,D_7}}(D,r_1^7)$	D	$C_{4\psi_{10,D_7}}(D,r_2^7)$	D	$C_{4\psi_{10,D_7}}(D,r_3^7)$
-1	-120	-1/2	4	-1/8	1
-2	-2144	-3/2	720	-9/8	9
-3	-6480	-5/2	4320	-17/8	-240
-4	-4800	-7/2	6720	-25/8	1705
-5	11760	-9/2	36	-33/8	-6480
-6	28800	-11/2	-21840	-41/8	16320
-7	43680	-13/2	-47520	-49/8	-33551
-8	43264	-15/2	-25920	-57/8	65520
-9	-59400	-17/2	-960	-65/8	-111360
-10	-103680	-19/2	29040	-73/8	145200
-11	-76560	-21/2	181440	-81/8	-174879
-12	-259200	-23/2	156480	-89/8	267600
-13	-46320	-25/2	6820	-97/8	-357360
-14	268800	-27/2	181440	-105/8	272160
-15	-61920	-29/2	-142560	-113/8	-188640
-16	360960	-31/2	-670080	-121/8	373561
-17	814560	-33/2	-25920	-129/8	-422640
-18	-19296	-35/2	-325920	-137/8	-46080
-19	157200	-37/2	-859680	-145/8	428160
-20	470400	-39/2	763200	-153/8	-118800
-21	-1360800	-41/2	65280	-161/8	-134400
-22	-873600	-43/2	-357840	-169/8	-949031
-23	550560	-45/2	2138400	-177/8	2093040
-24	-2165760	-47/2	954240	-185/8	-775200
-25	-954600	-49/2	-134204	-193/8	-119280
-26	1140480	-51/2	1231200	-201/8	-2686320
-27	-1632960	-53/2	151200	-209/8	3830640
-28	1747200	-55/2	-1596480	-217/8	-789600
-29	3080880	-57/2	262080	-225/8	15345
-30	-1036800	-59/2	-1094640	-233/8	-2990640
-31	1915200	-61/2	-5084640	-241/8	4364400
-32	3352576	-63/2	60480	-249/8	-460080
-33	1774080	-65/2	-445440	-257/8	-3355200
-34	514560	-67/2	-2008080	-265/8	-1864320
-35	-1226400	-69/2	2332800	-273/8	6118560
-36	-2376000	-71/2	-2723520	-281/8	2936880
-37	-4286160	-73/2	580800	-289/8	-8325791
-38	1161600	-75/2	5727600	-297/8	-1632960
-39	-2475360	-77/2	3991680	-305/8	4133280
-40	-6359040	-79/2	4798080	-313/8	9286080
-41	-3103200	-81/2	-699516	-321/8	-11722320
-42	-4354560	-83/2	2604240	-329/8	829920
-43	6051120	-85/2	4950720	-337/8	848160
-44	-3062400	-87/2	-1581120	-345/8	6697440
-45	105840	-89/2	1070400	-353/8	-15909120
-46	6259200	-91/2	-7355040	-361/8	8185321

-47	-2938560	-93/2	-8320320	-369/8	8078400
-48	19491840	-95/2	2516160	-377/8	12790560
-49	4026120	-97/2	-1429440	-385/8	-25243680
-50	-3655520	-99/2	-10810800	-393/8	-6499440
-51	8010720	-101/2	-16593120	-401/8	7261200
-52	-1852800	-103/2	3977280	-409/8	13657440
-53	12096720	-105/2	1088640	-417/8	-6283440
-54	7257600	-107/2	-1398000	-425/8	-1909200
-55	-12951840	-109/2	18208800	-433/8	4599600
-56	-20213760	-111/2	-20525760	-441/8	-301959
-57	-3369600	-113/2	-754560	-449/8	-13457520
-58	3421440	-115/2	29445600	-457/8	-4219200
-59	-16699440	-117/2	-427680	-465/8	32296320
-60	-2476800	-119/2	21275520	-473/8	-6850800
-61	-7595280	-121/2	1494244	-481/8	-10126560
-62	-26803200	-123/2	4548960	-489/8	-26658000
-63	21621600	-125/2	20865600	-497/8	23335200
-64	1167360	-127/2	-1766400	-505/8	4792320
-65	-35810400	-129/2	-1690560	-513/8	16511040
-66	13893120	-131/2	-28391760	-521/8	-25002720
-67	12306000	-133/2	10825920	-529/8	12206929
-68	32582400	-135/2	-6531840	-537/8	-39625200
-69	21752640	-137/2	-184320	-545/8	27769440
-70	-13036800	-139/2	9478800	-553/8	11689440
-71	12166560	-141/2	-71383680	-561/8	32345280
-72	389376	-143/2	-13041600	-569/8	-12983520
-73	43661760	-145/2	1712640	-577/8	-20004960
-74	20632320	-147/2	45360	-585/8	-55123200
-75	-11048400	-149/2	-2147040	-593/8	8774880
-76	6288000	-151/2	-17501760	-601/8	10290480
-77	-20690880	-153/2	-475200	-609/8	65681280
-78	30528000	-155/2	-6619200	-617/8	17979600
-79	-35520960	-157/2	13301280	-625/8	-40592975
-80	-35374080	-159/2	33929280	-633/8	-22600080
-81	6288840	-161/2	-537600	-641/8	-18126480
-82	-34990080	-163/2	-13410960	-649/8	34416240
-83	17445360	-165/2	79885440	-657/8	1306800
-84	-54432000	-167/2	22975680	-665/8	-14209440
-85	-34049760	-169/2	-3796124	-673/8	-7112400
-86	-14313600	-171/2	261360	-681/8	27624240
-87	-41110560	-173/2	19712160	-689/8	-65909760
-88	65694720	-175/2	53457600	-697/8	71929440
-89	18082560	-177/2	8372160	-705/8	31416480
-90	-51321600	-179/2	45013680	-713/8	27870240
-91	-20573280	-181/2	-85445280	-721/8	-144762240
-92	22022400	-183/2	-40737600	-729/8	-45663831
-93	-18597600	-185/2	-3100800	-737/8	52764720
-94	38169600	-187/2	-42857760	-745/8	146311200
-95	132571680	-189/2	45722880	-753/8	-56732400

-96	-7004160	-191/2	-69475200	-761/8	63691200
-97	23990880	-193/2	-477120	-769/8	-78000240
-98	71933344	-195/2	22783680	-777/8	-38626560
-99	-689040	-197/2	-97644960	-785/8	-81925920
-100	-38184000	-199/2	-96212160	-793/8	21048960
-121	-83479560				
-169	24773400				
-225	-472527000				
-289	999094920				
-361	-1576502280				
-441	1992929400				
-529	-1464831480				
-625	1248657000				
-729	12107319480				
-841	-17870133480				
-961	9776678280				
-1089	-41322382200				
-1225	32027784600				
-1369	13544322840				
-1521	12262833000				
-1681	-23071708920				
-1849	-15585928200				
-2025	50027722200				
-2209	-294960418680				
-2401	169865855880				
-2601	494551985400				
-2809	141343224600				
-3025	-664079899800				
-3249	-780368628600				
-3481	536913532920				
-3721	-936128995560				
-3969	-210996870840				
-4225	197072397000				
-4489	1695804213240				
-4761	-725091582600				
-5041	-958470710520				
-5329	73113632520				
-5625	618085215000				
-5929	2800822717560				
-6241	-4204774713720				
-6561	1936995369480				

Table A.8: Fourier coefficients of ψ_{9,D_7} and ψ_{11,D_7}

D	$C_{\psi_{9,D_7}}(D,r_3^7)$	$C_{\psi_{11,D_7}}(D,r_3^7)$
-1/8	1	1
-9/8	237	-507
-17/8	1440	-15120
-25/8	245	-73075
-33/8	-1440	-166320
-41/8	-11520	-60480
-49/8	3353	53417
-57/8	-12960	1043280
-65/8	28800	604800
-73/8	15840	1829520
-81/8	17289	-967671
-89/8	-4320	589680
-97/8	36000	-7635600
-105/8	-100800	1058400
-113/8	-66240	-9465120
-121/8	70789	-693011
-129/8	-142560	-498960
-137/8	5760	15361920
-145/8	86400	-3628800
-153/8	108000	29710800
-161/8	80640	6773760
-169/8	149533	3205957
-177/8	-180000	24570000
-185/8	302400	-22226400
-193/8	59040	-48520080
-201/8	-298080	3311280
-209/8	-456480	-4823280
-217/8	141120	-128489760
-225/8	58065	37049025
-233/8	-465120	-22876560
-241/8	-47520	-11476080
-249/8	-154080	46433520
-257/8	385920	86667840
-265/8	-28800	-45964800
-273/8	544320	253380960
-281/8	684000	-57078000
-289/8	-331183	46974833
-297/8	-224640	205571520
-305/8	849600	31903200
-313/8	-11520	-296170560
-321/8	1208160	-50243760
-329/8	-1834560	136110240
-337/8	-941760	-207839520
-345/8	388800	-227253600
-353/8	-288000	-299980800
-361/8	-185059	-89291179

-369/8	-864000	118843200
-377/8	-72000	-143488800
-385/8	-705600	-360914400
-393/8	277920	605450160
-401/8	82080	19822320
-409/8	-515520	397867680
-417/8	2086560	-255059280
-425/8	2152800	632394000
-433/8	-1078560	449683920
-441/8	794661	-27082419
-449/8	-1162080	291982320
-457/8	840960	959152320
-465/8	2188800	-743299200
-473/8	2052000	-601246800
-481/8	-1650240	-270617760
-489/8	-1516320	729524880
-497/8	-2358720	-89540640
-505/8	1065600	-1294876800
-513/8	-2021760	-1289494080
-521/8	54720	-394117920
-529/8	75337	1334599033
-537/8	-2717280	-1683354960
-545/8	-561600	-922168800
-553/8	504000	794858400
-561/8	1192320	-884157120
-569/8	-5175360	932329440
-577/8	452160	1849569120
-585/8	2160000	-1188432000
-593/8	3360960	1438728480
-601/8	3610080	-801949680
-609/8	-1128960	2097325440
-617/8	3566880	495074160
-625/8	-1739975	2977455625
-633/8	-5404320	-547994160
-641/8	6564960	-785982960
-649/8	3615840	2549186640
-657/8	3754080	-927566640
-665/8	-3528000	-2249100000
-673/8	2988000	825022800
-681/8	-6078240	-964398960
-689/8	-1169280	2388597120
-697/8	8640	-2738020320
-705/8	-2462400	-4374064800
-713/8	-2508480	-1138868640
-721/8	0	-1257379200
-729/8	-1967787	2004363117
-737/8	-6798240	-94636080
-745/8	6379200	-4928666400
-753/8	1412640	-3712398480

-761/8	-6439680	396204480
-769/8	-3051360	4309305840
-777/8	-5080320	4714536960
-785/8	3585600	-4128213600
-793/8	2206080	1852381440
-841/8	-4547731	-1485406411
-961/8	2232929	-3708048991
-1089/8	16776993	351356577
-1225/8	821485	-3903447275
-1369/8	2432917	-22150780307
-1521/8	35439321	-1625420199
-1681/8	-1313719	-7474274599
-1849/8	-30251803	-45440958067
-2025/8	4235805	70712558325
-2209/8	15460753	64352048497
-2401/8	-37161551	-100320091151
-2601/8	-78490371	-23816240331
-2809/8	8725733	60469078877
-3025/8	17343305	50641778825
-3249/8	-43858983	45270627753
-3481/8	-100117739	436446886621
-3721/8	-47043059	619541397781
-3969/8	57970017	-51690081807
-4225/8	36635585	-234275307775
-4489/8	141523373	-297698067643
-4761/8	17854869	-676641709731
-5041/8	412584409	516872527849
-5329/8	226841833	1389638711737
-5625/8	-412374075	-1509570001875
-5929/8	237355517	-37018568587
-6241/8	-453151759	3549597014801
-6561/8	-647274159	-934612680879

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