Jacobi forms and linear operators

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Jacobi forms

Automorphic forms are a central object in the study of Number Theory. They are *complex-valued* functions that satisfy a transformation property similar to *periodicity*.

Amongst various types of automorphic forms, **Jacobi forms** are an elegant *intermediate* between elliptic modular forms and Siegel modular forms.

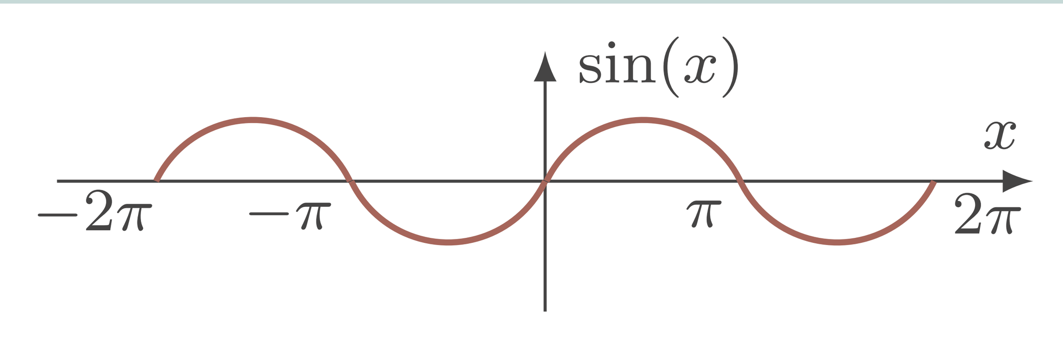

Figure 1: periodicity of sin

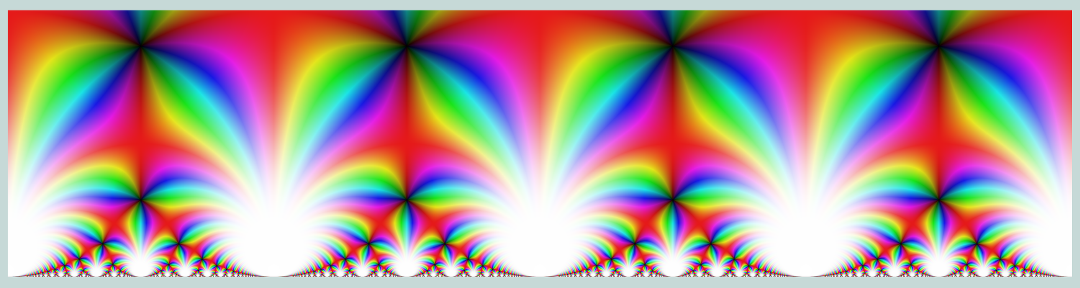

Figure 2: the j-function

Level raising operators

Linear operators give structure to a vector space. We defined the **level raising operators** $V(l)$, for any positive integer l:

$$\varphi|V(l)(\tau,\mathfrak{z}) = l^{\frac{k}{2}-1} \sum_{\substack{M\in \mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{M}_2(\mathbb{Z})\\ \det(M)=l}} (\varphi|_{k,\underline{L}}M)\,|U(\sqrt{l})(\tau,\mathfrak{z}),$$

where $|_{k,\underline{L}}$ is the action of $\mathrm{GL}_2(\mathbb{Z})$ and $\varphi|U(l)(\tau,\mathfrak{z}) = \varphi(\tau, l\mathfrak{z})$. Level raising means that $V(l)$ maps a Jacobi form in $J_{k,\underline{L}}$ to one in $J_{k,\underline{L}'}$, where $\underline{L}'$ has *level* equal to $l\cdot\mathrm{lev}(\underline{L})$. Any periodic function is uniquely determined by its **Fourier expansion**. For example, if the Fourier expansion of φ is

$$\varphi(\tau,\mathfrak{z}) = \sum_{r\in L^{\#}} \sum_{\substack{n\in\mathbb{Z}\\ n\geq\beta(r)}} c(n,r)e(n\tau+\beta(r,\mathfrak{z})),$$

then

$$\varphi|V(l)(\tau,\mathfrak{z}) = \sum_{\substack{n\in\mathbb{Z},r'\in L'^{\#}\\ n\geq\beta'(r')}} \sum_{\substack{a|(n,l)\\ \frac{r'}{a}\in L'^{\#}}} a^{k-1}c\left(\frac{nl}{a^2},\frac{lr'}{a}\right)e(n\tau+\beta'(r',\mathfrak{z})),$$

where $L^{\#}$ is the *dual* of $\underline{L}$ and $e(x) = e^{2\pi i x}$.

Properties

They are parametrized by *weight* (an integer) and *index* (a positive-definite, symmetric, integral lattice). For fixed weight k and index $\underline{L} = (L,\beta)$, they form a *vector space* over $\mathbb{C}$, denoted by $J_{k,\underline{L}}$.

They are functions of *two variables*: τ in the *upper half-plane* $\mathfrak{H} = \{z\in\mathbb{C} : \Im(z) > 0\}$ and $\mathfrak{z} = (z_1,\ldots,z_n)$ in $\mathbb{C}^n$ (where n is the *rank* of L).

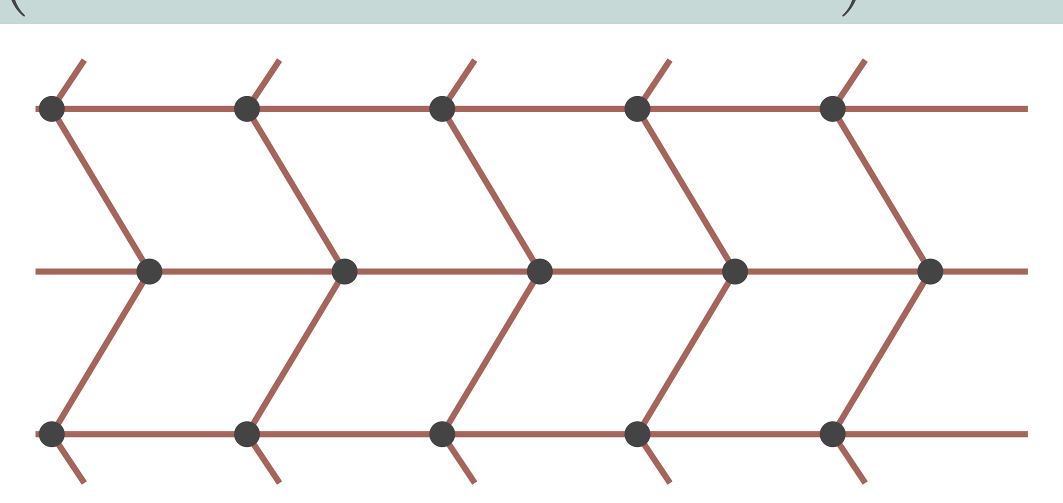

Figure 3: a lattice in $2D$

Commutative properties of V

We described the **commutative properties** of these operators:

$$\varphi|V(l)|V(l') = \sum_{d|\gcd(l,l')} d^{k-1}\varphi|V\left(\frac{ll'}{d^2}\right)|U(d),$$

which shows that they commute and shows how they behave when applied consecutively. Furthermore, we showed that they commute with two other important operators:

$$\varphi|U(l)|V(l') = \varphi|V(l')|U(l),$$

and

$$\varphi|T(l)|V(l') = \varphi|V(l')|T(l),$$

where $U(\cdot)$ is as above and $T(l)$ is the l-th Hecke operator, which is one of the most useful operators defined on modular forms. Similar relations hold for the equivalent operators defined on different types of automorphic forms. This suggests there exist explicit correspondences between spaces of Jacobi forms and spaces of different types of automorphic forms.

Future work

The operators $U(l)$ and $V(l)$ are a first step in the direction of a *theory of newforms* for Jacobi forms. **Newforms** are those forms in a fixed space $J_{k,\underline{L}}$ which don't 'come from' a space with lower level (if they do, we call them **oldforms**). They form part of a *basis* for Jacobi forms and give us a way of decreasing the number of things we have to work with.