Level raising operators for Jacobi forms of lattice index

Andreea Mocanu

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Euler International Mathematical Institute
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University of Nottingham
UK | CHINA | MALAYSIA

Notation

- the *weight* of a Jacobi form will be k in $\mathbb{N}$ and the *index* $\underline{L} = (L, \beta)$:
  - L is a *free, finite rank* $\mathbb{Z}$-module
  - $\beta : L \times L \to \mathbb{Z}$ is a $\mathbb{Z}$-bilinear form which is *symmetric*, *positive-definite*, *even*
- the *rank* of $\underline{L}$ is $\mathrm{rk}(\underline{L})$, where $L \simeq \mathbb{Z}^{\mathrm{rk}(\underline{L})}$
- set $\beta(\lambda) := \frac{1}{2}\beta(\lambda, \lambda)$
- the *dual lattice* of $\underline{L}$:
  $L^{\#} := \{t \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(\lambda, t) \in \mathbb{Z}$ for all λ in $L\}$
- the *determinant* of $\underline{L}$ is $\det(\underline{L}) := |L^{\#}/L|$
- the *level* of $\underline{L}$:
  $\mathrm{lev}(\underline{L}) := \min\{N \in \mathbb{N} : N\beta(t) \in \mathbb{Z}$ for all t in $L^{\#}\}$

Jacobi forms of lattice index

Definition

A function ϕ in $\mathrm{Hol}(\mathfrak{H} \times (L \otimes \mathbb{C}) \to \mathbb{C})$ is called a Jacobi form of weight k and index $\underline{L}$ if:

1. for every (A, h) in $J^{\underline{L}} := \mathrm{SL}_2(\mathbb{Z}) \ltimes L^2$, we have $\phi|_{k,\underline{L}}(A, h) = \phi$, where

$$\phi|_{k,\underline{L}}\left(A,(\lambda,y)\right)(\tau,z) := \phi\left(A\tau, \frac{z+\lambda\tau+\mu}{c\tau+d}\right)(c\tau+d)^{-k} \times e\left(\frac{-c\beta(z+\lambda\tau+\mu)}{c\tau+d} + \tau\beta(\lambda) + \beta(\lambda,z)\right)$$

2. ϕ has a Fourier expansion of the form

$$\phi(\tau,z) = \sum_{\substack{D\in\mathbb{Q}_{\leq 0}, t\in L^{\#} \\ \beta(t)-D\in\mathbb{Z}}} C(D,t)e\left((\beta(t)-D)\tau + \beta(t,z)\right).$$

- for fixed k and $\underline{L}$, denote the $\mathbb{C}$-vector space of all such functions by $J_{k,\underline{L}}$

- *Jacobi cusp forms* have the following type of Fourier expansion:

$$\phi(\tau, z) = \sum_{\substack{D \in \mathbb{Q}_{<0}, t \in L^{\#} \\ \beta(t) - D \in \mathbb{Z}}} C(D, t) e\left((\beta(t) - D)\tau + \beta(t, z)\right)$$

- denote the subspace of cusp forms of weight k and index $\underline{L}$ by $S_{k,\underline{L}}$
- the *isotropy set* of $\underline{L}$ is $\mathrm{Iso}(D_{\underline{L}}) := \{r \in L^{\#}/L : \beta(r) = 0\}$
- define $J_{\infty}^{\underline{L}} := \{(\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), (0, \mu)) : n \in \mathbb{Z}, \mu \in L\}$

Definition

For every r in $\mathrm{Iso}(D_{\underline{L}})$, let $g_{\underline{L},r}(\tau,z) := e(\beta(r)\tau + \beta(r,z))$ and define the Eisenstein series of weight k and index $\underline{L}$ associated to r as

$$E_{k,\underline{L},r}(\tau,z) := \frac{1}{2}\sum_{\gamma\in J^{\underline{L}}_{\infty}\backslash J^{\underline{L}}} g_{\underline{L},r}|_{k,\underline{L}}\gamma(\tau,z).$$

- defined by Ajouz; it is absolutely and uniformly convergent on compact subsets of $\mathfrak{H}\times(L\otimes\mathbb{C})$ for $k > \frac{\mathrm{rk}(\underline{L})}{2}+2$
- it is an element of $J_{k,\underline{L}}$ and it is orthogonal to cusp forms

Definition (Twisted Eisenstein series)

Let N_r denote the order of r in $L^{\#}/L$. For every primitive Dirichlet character modulo F ($F \mid N_r$), define the *twisted* Eisenstein series

$$E_{k,\underline{L},r,\chi}(\tau, z) := \sum_{d \in \mathbb{Z}_{N_r}^{\times}} \chi(d) E_{k,\underline{L},dr}(\tau, z)$$

- set $J_{k,\underline{L}}^{Eis} := \mathrm{Span}\{E_{k,\underline{L},r} : r \in \mathrm{Iso}(D_{\underline{L}})\}$
- Ajouz showed that the $E_{k,\underline{L},r,\chi}$ form a *basis of eigenforms* of $J_{k,\underline{L}}^{Eis}$ with eigenvalues given by *twisted divisor sums*

Level raising operators

- Eichler & Zagier use level raising operators as a main tool to develop a theory of newforms
- for lattice index, Ajouz showed that e.g. if

$$\underline{L} \simeq (\mathbb{Z}, (x, y) \mapsto \det(\underline{L})xy)$$

  then

$$J_{k,\underline{L}} \simeq \mathfrak{M}^{-}_{2k-1-\mathsf{rk}(\underline{L})}(\mathsf{lev}(\underline{L})/4)$$

- the notion of newforms is usually applied to cusp forms, but Eichler & Zagier study the action of level raising operators on Eisenstein series
- Skoruppa & Zagier (1988) use this to compute a trace formula for $J_{k,m}$

$$J_{k,\underline{L}} = S_{k,\underline{L}} \oplus J^{Eis}_{k,\underline{L}}$$

Definition (Isometry)

Let $\underline{L}_1 = (L_1, \beta_1)$ and $\underline{L}_2 = (L_2, \beta_2)$ be two lattices. An injective linear map $\sigma : L_1 \otimes \mathbb{Q} \to L_2 \otimes \mathbb{Q}$ such that $\beta_2 \circ \sigma = \beta_1$ and $\sigma \underline{L}_1 \subseteq \underline{L}_2$ is called an isometry of $\underline{L}_1$ into $\underline{L}_2$.

Definition (Level raising operator)

Let $\underline{L}_1$ and $\underline{L}_2$ be two positive-definite, even lattices. For every isometry σ of $\underline{L}_1$ into $\underline{L}_2$, define a linear operator $U(\sigma) : J_{k,\underline{L}_2} \to \mathrm{Hol}(\mathfrak{H} \times (L_1 \otimes \mathbb{C}) \to \mathbb{C})$,

$$\phi | U(\sigma)(\tau, z) := \phi(\tau, \sigma(z)).$$

- take $\underline{L}_m = (\mathbb{Z}, (x, y) \mapsto 2mxy)$; then $J_{k,\underline{L}_m} = J_{k,m}$
- $\sigma_l : \mathbb{Q} \to \mathbb{Q}, \sigma_l(x) = lx$ is an isometry of $\underline{L}_{ml^2}$ into $\underline{L}_m$
- $U(\sigma_l) = U_l$, with $U_l : J_{k,m} \to J_{k,ml^2}, \phi | U_l(\tau, z) = \phi(\tau, lz)$

Theorem

When $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$ as modules over $\mathbb{Q}$, the operator $U(\sigma)$ maps $J_{k,\underline{L}_2}$ to $J_{k,\underline{L}_1}$. If ϕ in $J_{k,\underline{L}_2}$ has a Fourier expansion of the type

$$\phi(\tau, z_2) = \sum_{\substack{D \in \mathbb{Q}_{\leq 0}, r \in L_2^{\#} \\ D - \beta_2(r) \in \mathbb{Z}}} C(D, r) e\left((\beta_2(r) - D)\tau + \beta_2(r, z_2)\right),$$

then $\phi|U(\sigma)$ has the following Fourier expansion:

$$\phi|U(\sigma)(\tau, z_1) = \sum_{\substack{D \in \mathbb{Q}_{\leq 0}, x \in L_1^{\#} \\ D - \beta_1(x) \in \mathbb{Z}}} C(D, \sigma(x)) e\left((\beta_1(x) - D)\tau + \beta_1(x, z_1)\right),$$

with the convention that $C(D, \sigma(x)) = 0$ unless $\sigma(x) \in L_2^{\#}$.

- we need $L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q}$ to compute the Fourier expansion of $\phi|U(\sigma)$; change of variable: $x = \sigma^{-1}(r)$

- the definition of the dual lattice implies that $\sigma^{-1}(L_2^{\#}) \leq L_1^{\#}$ and hence $\text{lev}(\underline{L}_2) \mid \text{lev}(\underline{L}_1)$
- it follows that $U(\sigma)$ *raises the level*
- let M be a matrix of σ and set $\det(\sigma) := |\det(M)|$; it is easy to prove that $\det(\underline{L}_1) = \det(\sigma)^2 \det(\underline{L}_2)$
- $\det(\underline{L})$ and $\text{lev}(\underline{L})$ share the same set of prime divisors; we can pinpoint the set of prime divisors of $\frac{\text{lev}(\underline{L}_1)}{\text{lev}(\underline{L}_2)}$ to $p \mid \det(\sigma)$ such that $p \nmid \text{lev}(\underline{L}_2)$, plus possibly some $p \mid \text{lev}(\underline{L}_2)$
- in the scalar case, the level is raised by $\det(\sigma)^2$; in order to modify the level by an arbitrary positive integer, the Hecke-type operators V_l are introduced by Eichler and Zagier

Example (Counter-example)

Consider the positive-definite, even lattices

$$\underline{L}_1 = \left(\mathbb{Z}^2, \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)\right) \mapsto 24xs + 3ys + 3xt + 18y^2\right),$$
$$\underline{L}_2 = \left(\mathbb{Z}^2, \left(\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \left(\begin{smallmatrix} s \\ t \end{smallmatrix}\right)\right) \mapsto 24xs + ys + xt + 2yt\right).$$

There exists an isometry σ_{3y} of $\underline{L}_1$ into $\underline{L}_2$, mapping $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ to $\left(\begin{smallmatrix} x \\ 3y \end{smallmatrix}\right)$. It gives rise to a linear operator $U(\sigma_{3y})$ mapping $J_{k,\underline{L}_2}$ to $J_{k,\underline{L}_1}$. Using Sage, one can check that lev$(\underline{L}_1) = 141$ and lev$(\underline{L}_2) = 47$.

- $(\sigma(L_1), \beta_2)$ is a sublattice of $\underline{L}_2$ and $\underline{L}_1 \simeq (\sigma(L_1), \beta_2)$
- conversely, any sublattice (M, β_2) of $\underline{L}_2$ gives rise to an isometry of (M, β_2) into $\underline{L}_2$
- given $\underline{L}$, we want the classification of *overlattices* of $\underline{L}$

Proposition (Nikulin, 1980)

Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice. Then there is a one-to-one correspondence between overlattices of $\underline{L}$ and isotropic subgroups of $D_{\underline{L}}$. For every such overlattice $\underline{L}' = (L', \beta)$, the correspondence is given by

$$\underline{L}' \mapsto L'/L.$$

- $L \hookrightarrow L' \hookrightarrow L'^{\#} \hookrightarrow L^{\#}$

Newforms

- let $\mathcal{I}_{\underline{L}}$ denote the set of isotropic subgroups of $L^{\#}/L$
- the orthogonal group of $\underline{L}$ acts on $\mathcal{I}_{\underline{L}}$ from the right via

$$(\alpha, I) \mapsto \tilde{\alpha}(I).$$

- two overlattices $\underline{L}'$ and $\underline{L}''$ of $\underline{L}$ are isomorphic if and only if $[L'/L] = [L''/L]$ in $O(\underline{L}) \setminus \mathcal{I}_{\underline{L}}$
- for every element I in $\mathcal{I}_{\underline{L}}$, set $\underline{L}_I := (L + I, \beta)$ and let ι_I denote the inclusion map between L and L_I and set $U_I := U(\iota_I)$

Definition

Let $\underline{L}$ be a positive-definite, even lattice. Define the space of oldfroms of weight k and index $\underline{L}$ with respect to isometries as

$$J_{k,\underline{L}}^{\text{old,iso}} := \sum_{\substack{I \in O(\underline{L}) \setminus \mathcal{I}_{\underline{L}} \\ I \neq 0}} J_{k,\underline{L}_I} | U_I$$

- it is easy to show that U maps Eisenstein series to Eisenstein series and cusp forms to cusp forms
- given $\underline{L}$, fix r in $\mathrm{Iso}(D_{\underline{L}})$ and $F \mid N_r$
- write $N_r = N_0 \prod_{p|F} p^{v_p(N_r)}$
- for every divisor f of N_0, set $r_f := fFr$

Theorem

If χ is a primitive Dirichlet character modulo F for some $F \mid N_r$ such that $F \neq N_r$, then $E_{k,\underline{L},r,\chi}$ is an oldfrom. More precisely,

$$E_{k,\underline{L},r,\chi} = \chi(N_0) \sum_{f|N_0} \mu(f) E_{k,\underline{L}_{\langle r_f \rangle},N_0 r,\chi} | U_{\langle r_f \rangle}.$$

- this was shown for vector-valued modular forms by Schwagenscheidt (2018)

$$E_{k,\underline{L},r,\chi}(\tau, z) = \sum_{d \in \mathbb{Z}_{N_r}^{\times}} \chi(d) E_{k,\underline{L},dr}(\tau, z)$$

To do

- we want to classify old Jacobi forms obtained in this way explicitly
- we need to compute $\frac{\operatorname{lev}(\underline{L}_I)}{\operatorname{lev}(\underline{L})}$
- define V_l?

Thank you!

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Questions?