Level raising operators for Jacobi forms of lattice index

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the weight of a Jacobi form will be \( k \) in \( \mathbb{N} \) and the index \( \underline{L} = (L, \beta) \):

- \( L \) is a free, finite rank \( \mathbb{Z} \)-module
- \( \beta : L \times L \to \mathbb{Z} \) is a \( \mathbb{Z} \)-bilinear form which is symmetric, positive-definite, even

the rank of \( \underline{L} \) is \( \text{rk}(L) \), where \( L \sim \mathbb{Z}^{\text{rk}(L)} \)

set \( \beta(\lambda) := \frac{1}{2} \beta(\lambda, \lambda) \)

the dual lattice of \( \underline{L} \):
\[
L^\# := \{ t \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(\lambda, t) \in \mathbb{Z} \text{ for all } \lambda \text{ in } L \}
\]

the determinant of \( \underline{L} \) is \( \det(\underline{L}) := \left| L^\# / L \right| \)

the level of \( \underline{L} \):
\[
\text{lev}(\underline{L}) := \min \{ N \in \mathbb{N} : N \beta(t) \in \mathbb{Z} \text{ for all } t \text{ in } L^\# \} 
\]
A function \( \phi \) in \( \text{Hol}(\mathcal{H} \times (L \otimes \mathbb{C}) \to \mathbb{C}) \) is called a Jacobi form of weight \( k \) and index \( L \) if:

1. for every \((A, h)\) in \( J_L := \text{SL}_2(\mathbb{Z}) \rtimes L^2 \), we have \( \phi|_{k,L}(A, h) = \phi \), where

\[
\phi|_{k,L} (A, (\lambda, y)) (\tau, z) := \phi \left( A\tau, \frac{z + \lambda \tau + \mu}{c\tau + d} \right) (c\tau + d)^{-k} \\
\times e \left( \frac{-c\beta(z + \lambda \tau + \mu)}{c\tau + d} + \tau \beta(\lambda) + \beta(\lambda, z) \right)
\]

2. \( \phi \) has a Fourier expansion of the form

\[
\phi(\tau, z) = \sum_{\substack{D \in \mathbb{Q} \leq 0, \, t \in L^\# \\ \beta(t) - D \in \mathbb{Z}}} C(D, t) e \left( (\beta(t) - D)\tau + \beta(t, z) \right).
\]

- for fixed \( k \) and \( L \), denote the \( \mathbb{C} \)-vector space of all such functions by \( J_{k,L} \)
Jacobi cusp forms have the following type of Fourier expansion:

\[ \phi(\tau, z) = \sum_{D \in \mathbb{Q}_{<0}, t \in L^\#} C(D, t) e \left( (\beta(t) - D)\tau + \beta(t, z) \right) \]

- denote the subspace of cusp forms of weight \( k \) and index \( \underline{L} \) by \( S_{k, \underline{L}} \)
- the isotropy set of \( \underline{L} \) is \( \text{Iso}(D_{\underline{L}}) := \{ r \in L^# / L : \beta(r) = 0 \} \)
- define \( J_{\infty}^{\underline{L}} := \{ ((\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, (0, \mu)) : n \in \mathbb{Z}, \mu \in L \} \)
Definition

For every $r$ in $\text{Iso}(D_L)$, let $g_{L,r}(\tau, z) := e(\beta(r)\tau + \beta(r, z))$ and define the Eisenstein series of weight $k$ and index $L$ associated to $r$ as

$$E_{k, L, r}(\tau, z) := \frac{1}{2} \sum_{\gamma \in J^L_{\infty} \setminus J^L} g_{L, r}\mid_{k, L} \gamma(\tau, z).$$

- defined by Ajouz; it is absolutely and uniformly convergent on compact subsets of $\mathfrak{H} \times (L \otimes \mathbb{C})$ for $k > \frac{\text{rk}(L)}{2} + 2$
- it is an element of $J_{k, L}$ and it is orthogonal to cusp forms
Definition (Twisted Eisenstein series)

Let $N_r$ denote the order of $r$ in $L^\# / L$. For every primitive Dirichlet character modulo $F$ ($F \mid N_r$), define the twisted Eisenstein series

$$E_{k,L,r,\chi}(\tau, z) := \sum_{d \in \mathbb{Z}^\times_{N_r}} \chi(d) E_{k,L,dr}(\tau, z)$$

set $J_{k,L}^{Eis} := \text{Span}\{E_{k,L,r} : r \in \text{Iso}(D_L)\}$

Ajouz showed that the $E_{k,L,r,\chi}$ form a basis of eigenforms of $J_{k,L}^{Eis}$ with eigenvalues given by twisted divisor sums.
Level raising operators

- Eichler & Zagier use level raising operators as a main tool to develop a theory of newforms.
- For lattice index, Ajouz showed that e.g. if \( L \cong (\mathbb{Z}, (x,y) \mapsto \det(L)xy) \)
  then
  \[
  J_{k,L} \cong \mathcal{M}_{2k-1-rk(L)}(lev(L)/4)
  \]
- The notion of newforms is usually applied to cusp forms, but Eichler & Zagier study the action of level raising operators on Eisenstein series.
- Skoruppa & Zagier (1988) use this to compute a trace formula for \( J_{k,m} \)
  \[
  J_{k,L} = S_{k,L} \oplus J_{k,L}^{Eis}
  \]
Isometries

Definition (Isometry)
Let $L_1 = (L_1, \beta_1)$ and $L_2 = (L_2, \beta_2)$ be two lattices. An injective linear map $\sigma : L_1 \otimes \mathbb{Q} \to L_2 \otimes \mathbb{Q}$ such that $\beta_2 \circ \sigma = \beta_1$ and $\sigma L_1 \subseteq L_2$ is called an isometry of $L_1$ into $L_2$.

Definition (Level raising operator)
Let $L_1$ and $L_2$ be two positive-definite, even lattices. For every isometry $\sigma$ of $L_1$ into $L_2$, define a linear operator $U(\sigma) : J_{k,L_2} \to \text{Hol}(\mathcal{H} \times (L_1 \otimes \mathbb{C}) \to \mathbb{C})$,

$$\phi|U(\sigma)(\tau, z) := \phi(\tau, \sigma(z)).$$

- take $L_m = (\mathbb{Z}, (x, y) \mapsto 2mxy)$; then $J_{k,L_m} = J_{k,m}$
- $\sigma_l : \mathbb{Q} \to \mathbb{Q}, \sigma_l(x) = lx$ is an isometry of $L_{ml^2}$ into $L_m$
- $U(\sigma_l) = U_l$, with $U_l : J_{k,m} \to J_{k,ml^2}, \phi|U_l(\tau, z) = \phi(\tau, lz)$
Theorem

When \( L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q} \) as modules over \( \mathbb{Q} \), the operator \( U(\sigma) \) maps \( J_{k, L_2} \) to \( J_{k, L_1} \). If \( \phi \) in \( J_{k, L_2} \) has a Fourier expansion of the type

\[
\phi(\tau, z_2) = \sum_{D \in \mathbb{Q}_{\leq 0}, r \in L_2^\#} C(D, r) e \left( (\beta_2(r) - D) \tau + \beta_2(r, z_2) \right),
\]

then \( \phi|U(\sigma) \) has the following Fourier expansion:

\[
\phi|U(\sigma)(\tau, z_1) = \sum_{D \in \mathbb{Q}_{\leq 0}, x \in L_1^\#} C(D, \sigma(x)) e \left( (\beta_1(x) - D) \tau + \beta_1(x, z_1) \right),
\]

with the convention that \( C(D, \sigma(x)) = 0 \) unless \( \sigma(x) \in L_2^\# \).

- we need \( L_1 \otimes \mathbb{Q} \simeq L_2 \otimes \mathbb{Q} \) to compute the Fourier expansion of \( \phi|U(\sigma) \); change of variable: \( x = \sigma^{-1}(r) \)
the definition of the dual lattice implies that $\sigma^{-1}(L_2^\#) \leq L_1^\#$ and hence $\text{lev}(L_2) \mid \text{lev}(L_1)$

it follows that $U(\sigma)$ raises the level

let $M$ be a matrix of $\sigma$ and set $\det(\sigma) := |\det(M)|$; it is easy to prove that $\det(L_1) = \det(\sigma)^2 \det(L_2)$

$\det(L)$ and $\text{lev}(L)$ share the same set of prime divisors; we can pinpoint the set of prime divisors of $\frac{\text{lev}(L_1)}{\text{lev}(L_2)}$ to $p \mid \det(\sigma)$ such that $p \nmid \text{lev}(L_2)$, plus possibly some $p \mid \text{lev}(L_2)$

in the scalar case, the level is raised by $\det(\sigma)^2$; in order to modify the level by an arbitrary positive integer, the Hecke-type operators $V_l$ are introduced by Eichler and Zagier
Example (Counter-example)

Consider the positive-definite, even lattices

\[
L_1 = \left( \mathbb{Z}^2, \left( \begin{array}{c} x \\ y \end{array} \right), \left( \begin{array}{c} s \\ t \end{array} \right) \right) \mapsto 24xs + 3ys + 3xt + 18y^2, \\
L_2 = \left( \mathbb{Z}^2, \left( \begin{array}{c} x \\ y \end{array} \right), \left( \begin{array}{c} s \\ t \end{array} \right) \right) \mapsto 24xs + ys + xt + 2yt. 
\]

There exists an isometry \( \sigma_{3y} \) of \( L_1 \) into \( L_2 \), mapping \( \left( \begin{array}{c} x \\ y \end{array} \right) \) to \( \left( \begin{array}{c} x \\ 3y \end{array} \right) \). It gives rise to a linear operator \( U(\sigma_{3y}) \) mapping \( J_{k,L_2} \) to \( J_{k,L_1} \).

Using Sage, one can check that \( \text{lev}(L_1) = 141 \) and \( \text{lev}(L_2) = 47 \).
• $(\sigma(L_1), \beta_2)$ is a sublattice of $L_2$ and $L_1 \sim (\sigma(L_1), \beta_2)$
• conversely, any sublattice $(M, \beta_2)$ of $L_2$ gives rise to an isometry of $(M, \beta_2)$ into $L_2$
• given $L$, we want the classification of *overlattices* of $L$

Proposition (Nikulin, 1980)

Let $L = (L, \beta)$ be a positive-definite, even lattice. Then there is a one-to-one correspondence between overlattices of $L$ and isotropic subgroups of $D_L$. For every such overlattice $L' = (L', \beta)$, the correspondence is given by

$$L' \mapsto L'/L.$$ 

• $L \leftrightarrow L' \leftrightarrow L'\# \leftrightarrow L\#$
Newforms

- let $\mathcal{I}_L$ denote the set of isotropic subgroups of $L^#/L$
- the orthogonal group of $L$ acts on $\mathcal{I}_L$ from the right via
  \[(\alpha, I) \mapsto \tilde{\alpha}(I).\]
- two overlattices $L'$ and $L''$ of $L$ are isomorphic if and only if
  \[[L'/L] = [L''/L]\text{ in } O(L) \setminus \mathcal{I}_L\]
- for every element $I$ in $\mathcal{I}_L$, set $L_I := (L + I, \beta)$ and let $\iota_I$
  denote the inclusion map between $L$ and $L_I$ and set
  $U_I := U(\iota_I)$

**Definition**

Let $L$ be a positive-definite, even lattice. Define the space of oldforms of weight $k$ and index $L$ with respect to isometries as

\[J_{k,L}^{\text{old,iso}} := \sum_{I \in \mathcal{O}(L) \setminus \mathcal{I}_L, I \neq 0} J_{k,L_I} |U_I\]
it is easy to show that $U$ maps Eisenstein series to Eisenstein series and cusp forms to cusp forms

- given $L$, fix $r$ in $\text{Iso}(D_L)$ and $F \mid N_r$
- write $N_r = N_0 \prod_{p|F} p^{\nu_p(N_r)}$
- for every divisor $f$ of $N_0$, set $r_f := fFr$

**Theorem**

If $\chi$ is a primitive Dirichlet character modulo $F$ for some $F \mid N_r$ such that $F \neq N_r$, then $E_{k,L,r,\chi}$ is an oldfrom. More precisely,

$$E_{k,L,r,\chi} = \chi(N_0) \sum_{f \mid N_0} \mu(f) E_{k,L \langle r_f \rangle, N_0r, \chi} | U \langle r_f \rangle.$$

this was shown for vector-valued modular forms by Schwagenscheidt (2018)

$$E_{k,L,r,\chi}(\tau, z) = \sum_{d \in \mathbb{Z}_{N_r}^X} \chi(d) E_{k,L,dr}(\tau, z)$$
we want to classify old Jacobi forms obtained in this way explicitly
we need to compute \( \frac{\text{lev}(L_I)}{\text{lev}(L)} \)
define \( V_i \)?
we want to classify old Jacobi forms obtained in this way explicitly

we need to compute $\frac{\text{lev}(L_f)}{\text{lev}(L)}$

define $V_l$?

Questions?