Newform theory for Jacobi forms of lattice index

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1. Setup

- $e(x) = e^{2\pi i x}$, $\mathfrak{H} := \{z \in \mathbb{C} : \Im(z) > 0\}$, Γ denotes $\mathrm{S}L_2(\mathbb{Z})$ and

$$\Gamma_0(m) := \left\{ \begin{pmatrix} * & * \\ 0 \bmod m & * \end{pmatrix} \right\} \cap \Gamma$$

- fix k in $\mathbb{N}$ and a lattice $\underline{L} = (L, \beta)$ over $\mathbb{Z}$ which is
  - *positive-definite*: $\beta(\lambda,\lambda) > 0$ for all λ in L
  - *even*: $\beta(\lambda) := \frac{\beta(\lambda,\lambda)}{2} \in \mathbb{Z}$
- the *rank* of $\underline{L}$ is the rank of L as a $\mathbb{Z}$-module: $L \simeq \mathbb{Z}^{\mathrm{rk}(\underline{L})}$
- the *dual* of $\underline{L}$ is $L^{\#} := \{t \in L \otimes_{\mathbb{Z}} \mathbb{Q} : \beta(t,\lambda) \in \mathbb{Z}$ for all λ in $L\}$
- the *level* of $\underline{L}$ is $\mathrm{lev}(\underline{L}) := \min\{N \in \mathbb{N} : N\beta(t) \in \mathbb{Z}$ for all t in $L^{\#}\}$
- the integral *Jacobi group* $J^{\underline{L}} := \mathrm{S}L_2(\mathbb{Z})u \ltimes L^2$ acts $\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C})$:

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, (\lambda, \mu) \right] (\tau, \mathfrak{z}) := \left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z} + \tau\lambda + \mu}{c\tau + d} \right)$$

- $J^{\underline{L}}$ acts on holomorphic functions $\varphi : \mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C}$ via:

$$\begin{aligned} \varphi|_{k,\underline{L}}\gamma(\tau,\mathfrak{z}) :=& \varphi\left(\gamma(\tau,\mathfrak{z})\right)(c\tau + d)^{-k} \\ & \times e\left(\frac{-c\beta(\mathfrak{z} + \lambda\tau + \mu)}{c\tau + d} + \tau\beta(\lambda) + \beta(\lambda, \mathfrak{z}) \right) \end{aligned}$$

Definition

A function φ as above is a Jacobi form of weight k and index $\underline{L}$ if

1. $\varphi|_{k,\underline{L}}\gamma(\tau,\mathfrak{z}) = \varphi(\tau,\mathfrak{z})$ for all γ in $J^{\underline{L}}$ and
2. φ has a Fourier expansion of the form

$$\varphi(\tau,\mathfrak{z}) = \sum_{\substack{n\in\mathbb{Z},t\in L^{\#}\\ n\geq\beta(t)}} c_\varphi(n,t)e\left(n\tau+\beta(t,\mathfrak{z})\right).$$

- $J_{k,\underline{L}}$:= {space of φ as above}
- $c_\varphi(n,t)$ only depends on $t \bmod L$ and on $n-\beta(t)$, so we can write

$$\varphi(\tau,\mathfrak{z}) = \sum_{\substack{D\in\mathbb{Q}_{\leq 0},t\in L^{\#}\\ D\equiv\beta(t) \bmod \mathbb{Z}}} C_\varphi(D,t)e\left((\beta(t)-D)\tau+\beta(t,\mathfrak{z})\right)$$

  with $C_\varphi(D,t) := c_\varphi(\beta(t)-D,t)$
- $S_{k,\underline{L}} := \{\varphi\in J_{k,\underline{L}} : C_\varphi(0,t)=0$ for all $t\in L^{\#}$ such that $\beta(t)\in\mathbb{Z}\}$

- $g_{\underline{L},D,t}(\tau,\mathfrak{z}) := e\left((\beta(t)-D)\tau + \beta(t,\mathfrak{z})\right)$
- $J_{\infty}^{\underline{L}} := \{\left(\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right),(0,\mu)\right) : n\in\mathbb{Z}, \mu\in L\}$
- for every pair (D,t) such that $D\in\mathbb{Q}_{<0}$, $t\in L^{\#}$ and $\beta(t)\equiv D \bmod \mathbb{Z}$, define the *Poincaré series*

$$P_{k,\underline{L},D,t}(\tau,\mathfrak{z}) := \sum_{\gamma\in J_{\infty}^{\underline{L}}\backslash J^{\underline{L}}} g_{\underline{L},D,t}|_{k,\underline{L}}\gamma(\tau,\mathfrak{z})$$

  and, for every r in $L^{\#}$ such that $\beta(r)\in\mathbb{Z}$, define the *Eisenstein series*

$$E_{k,\underline{L},r}(\tau,\mathfrak{z}) := \frac{1}{2}\sum_{\gamma\in J_{\infty}^{\underline{L}}\backslash J^{\underline{L}}} g_{\underline{L},0,r}|_{k,\underline{L}}\gamma(\tau,\mathfrak{z})$$

- set $\mathrm{Iso}(\underline{L}) := \{r\in L^{\#}/L : \beta(r)\in\mathbb{Z}\}$ and $J_{k,\underline{L}}^{\mathrm{Eis}} := \{E_{k,\underline{L},r} : r\in\mathrm{Iso}(\underline{L})\}$

Theorem (M. 2017)

The Poincaré series span $S_{k,\underline{L}}$ *and*

$$J_{k,\underline{L}} = S_{k,\underline{L}} \oplus J^{\mathrm{Eis}}_{k,\underline{L}}.$$

- meaning:
  - $k > \mathrm{rk}(\underline{L}) + 2$ for Poincaré series and $k > \frac{\mathrm{rk}(\underline{L})}{2} + 2$ for Eisenstein series
  - $P_{k,\underline{L},D,t} \in S_{k,\underline{L}}$ and $E_{k,\underline{L},r} \in J_{k,\underline{L}}$
  - $\exists$ constants $\lambda_{k,\underline{L},D}$ in $\mathbb{C}$ such that $\langle \varphi, P_{k,\underline{L},D,t} \rangle = \lambda_{k,\underline{L},D} C_{\varphi}(D,t)$ for every φ in $S_{k,\underline{L}}$
  - $\langle J^{\mathrm{Eis}}_{k,\underline{L}}, S_{k,\underline{L}} \rangle = 0$
- Fourier expansions: $C_{P_{k,\underline{L},D,t}}(0,s) = 0$ and

$$\begin{aligned} C_{P_{k,\underline{L},D,t}}(G,s) :=& \delta_{\underline{L}}(D,t,G,s) + (-1)^k \delta_{\underline{L}}(D,-t,G,s) + \frac{2\pi i^k}{\det(\underline{L})^{\frac{1}{2}}} \\ & \times \left(\frac{G}{D}\right)^{\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} - \frac{1}{2}} \sum_{c \geq 1} c^{-\frac{\mathrm{rk}(\underline{L})}{2} - 1} J_{k - \frac{\mathrm{rk}(\underline{L})}{2} - 1}\left(\frac{4\pi (DG)^{\frac{1}{2}}}{c}\right) \\ & \times \Big(H_{\underline{L},c}(D,t,G,s) + (-1)^k H_{\underline{L},c}(D,-t,G,s) \Big), \end{aligned}$$

with $H_{\underline{L},c}(D,t,G,s)$ equal to

$$\sum_{d \in (\mathbb{Z}/c\mathbb{Z})^{\times}, \lambda \in L/cL} e_c\left(\beta(\lambda + t) - D)d^{-1} + (\beta(s) - G)d + \beta(s, \lambda + t)\right)$$

- the Dedekind η-function

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n\geq 1} (1-q^n)$$

is an elliptic modular form of weight $1/2$ for Γ; the Jacobi theta series

$$\vartheta(\tau, z) := \sum_{n\in\mathbb{Z}} \left(\tfrac{-4}{n}\right) e\left(\tau\frac{n^2}{8} + \frac{nz}{2}\right)$$

is a Jacobi form of weight $\frac{1}{2}$ and scalar index $\frac{1}{2}$

- for $2 \leq k \leq 8$, the function

$$\psi_{12-k,D_k}(\tau, \mathfrak{z}) := \eta(\tau)^{24-3k}\vartheta(\tau, z_1)\ldots\vartheta(\tau, z_k)$$

is an element of J_{12-k,D_k} ($\mathfrak{z} = (z_1, \ldots, z_k)$) and in fact an element of S_{12-k,D_k} for $k \leq 7$ (note that $D_3 = A_3$)

- define

$$\Theta(\tau, z_1, z_2) := \vartheta(\tau, z_1)\vartheta(\tau, z_2 - z_1)\vartheta(\tau, z_2)/\eta(\tau) \in J_{1,A_2}(v_\eta^8);$$

then

$$\begin{aligned}
\psi_{9,A_2}(\tau,\mathfrak{z}) &:= \eta^{16}(\tau)\Theta(\tau, z_1, z_2) \in S_{9,A_2} \\
\psi_{6,2A_2}(\tau,\mathfrak{z}) &:= \eta^{8}(\tau)\Theta(\tau, z_1, z_2)\Theta(\tau, z_3, z_4) \in S_{6,2A_2} \\
\psi_{3,3A_2}(\tau,\mathfrak{z}) &:= \Theta(\tau, z_1, z_2)\Theta(\tau, z_3, z_4)\Theta(\tau, z_5, z_6) \in J_{3,3A_2}
\end{aligned}$$

- Gritsenko–Skoruppa–Zagier (preprint): *Theta blocks*
- for every t in $L^\#/L$, define

$$\vartheta_{\underline{L},t}(\tau,\mathfrak{z}) := \sum_{\substack{s\in L^\# \\ s\equiv t \bmod L}} e\left(\beta(s)\tau + \beta(s,\mathfrak{z})\right);$$

  - if $\underline{L}$ is an even, unimodular lattice, then $\vartheta_{\underline{L},0}$ is an element of $J_{\frac{\mathbf{rk}(\underline{L})}{2},\underline{L}}$
  - in general, $E_{k,\underline{L},r} = \frac{1}{2}\left(\vartheta_{\underline{L},r} + (-1)^k\vartheta_{\underline{L},-r}\right) + \dots$

- Gritsenko (1988): introduces Jacobi forms of lattice index as *Fourier–Jacobi coefficients* of *orthogonal modular forms*
- embed $\underline{L}$ into a lattice of signature $(2, \mathrm{rk}(\underline{L})+2)$: $\underline{M} = H_1 \oplus \underline{L}(-1) \oplus H_2$
- write $Z = (\omega, \mathfrak{z}, \tau) \in \mathcal{H}(\underline{M})$ and embedd $J^{\underline{L}}$ into $O^+(\underline{M})$ as

$$O^J := \left\{ \mathrm{diag}(A^*, E_{\mathrm{rk}(\underline{L})}, A) \begin{pmatrix} 1 & 0 & \mu^t G & \beta(\lambda,\mu) & \beta(\mu) \\ 0 & 1 & \lambda^t G & \beta(\lambda) & 0 \\ 0 & 0 & E_{\mathrm{rk}(\underline{L})} & \lambda & \mu \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

  where $A^* = I(A^t)^{-1}I$ and $I = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$
- then $\varphi \in J_{k,\underline{L}} \iff \Phi(Z) := \varphi(\tau, \mathfrak{z})e(2\pi i\omega) \in M_k(O^J)$
- one can *lift* Jacobi forms to *reflective modular forms*
- some RMFs are the *automorphic discriminants* of moduli spaces (e.g. lattice polarized $K3$ surfaces - Gritsenko–Nikulin (1996))
- this allows for the construction of modular varieties (e.g. Gritsenko (2010): modular varieties of Calabi–Yau type of dim 4, 6 and 7 and Kodaira dim 0)
- if a RMF is a lift of a Jacobi form, then one obtains a simple formula for its Fourier coefficients at a 0-dim cusp; these determine the generators and relations of Lorentzian Kac–Moody algebras (Grisenko–Nikulin (1997))

Conjecture $\star$ (Ajouz, 2015)

If rk($\underline{L}$) is odd, then there exists a Hecke-equivariant isomorphism

$$J_{k,\underline{L}} \simeq \mathfrak{M}^{\varepsilon}_{2k-\mathrm{rk}(\underline{L})-1}(\mathrm{lev}(\underline{L})/4),$$

where $\varepsilon = -1$ if rk($\underline{L}$) $\equiv 1$ or $3 \bmod 8$ and $\varepsilon = 1$ otherwise.

- $W_m := \left(\begin{smallmatrix} 0 & -1 \\ m & 0 \end{smallmatrix}\right)$ and $M^{\varepsilon}_k(m) := \left\{ f \in M_k(m) : f|_k W_m = \varepsilon i^{-k} f \right\}$
- $f \in M^{\varepsilon}_k(m) \implies \Lambda_m(f,s) = \varepsilon \Lambda_m(f, k-s)$
- every Hecke eigenform f in $M_k(m)$ "comes from" a newform g in $M_k(n)$

$$\frac{L(f,s)}{L(g,s)} = \prod_{p \mid \frac{m}{n}} Q_p(s)$$

- if f is an eigenform for all Atkin–Lehner involutions, then every Q_p has a functional equation

$$Q_p(k-s) = \pm p^{-v_p(m/n)(k-2s)} Q_p(s)$$

- let $\mathfrak{M}_k(m)$ denote the subspace spanned by all f for which the sign in the above equation is $+$ for all $p \mid (m/n)$; set $\mathfrak{M}^{\varepsilon}_k(m) := \mathfrak{M}_k(m) \cap M^{\varepsilon}_k(m)$

- this was proved for $\mathsf{rk}(\underline{L}) = 1$ by Skoruppa–Zagier (1988) by using *trace formulas* and the theory of *newforms*
    - the result was used by Gross–Kohnen–Zagier (1987) to show that the classes of Heegner points on a modular curve $X_0(m)$ in the Mordell–Weil group of its Jacobian are the coefficients of a Jacobi form of weight 2 and index m

Goal

Study Ajouz's conjecture.

- intermediate goal: develop e theory of newforms, meaning

Given $J_{k,\underline{L}}$, how many of its elements 'come from' Jacobi forms of weight k and index $\underline{M}$ with $\mathrm{lev}(\underline{M}) \mid \mathrm{lev}(\underline{L})$?

- Conjecture $\star$:

$$J_{k,\underline{L}} \simeq \mathfrak{M}^{\pm 1}_{2k-1-\mathsf{rk}(\underline{L})}(\mathrm{lev}(\underline{L})/4),$$

- Sakata (2018):

$$S^{\mathrm{new},+}_{k,1}(m) \simeq S^{\mathrm{new}}_{k,m}$$

- defined by Ajouz (2015) for $(\ell, \mathrm{lev}(\underline{L})) = 1$:

$$T_0(\ell)\varphi := \ell^{k-\mathsf{rk}(\underline{L})-2} \sum_{\gamma \in J^{\underline{L}} \backslash J^{\underline{L}} \left(\begin{smallmatrix} 1/\ell & 0 \\ 0 & \ell \end{smallmatrix}\right) J^{\underline{L}}} \varphi|_{k,\underline{L}}\gamma$$

- then

$$T(\ell)\varphi := \begin{cases} \displaystyle\sum_{d^2|\ell} d^{2k-\mathsf{rk}(\underline{L})-3} T_0\left(\frac{\ell}{d^2}\right)\varphi, & \mathsf{rk}(\underline{L}) \text{ odd} \\ \displaystyle\sum_{\substack{s,d>0 \\ sd^2|\ell, s \text{ square-free}}} \chi_{\underline{L}}(s)(sd^2)^{k-\frac{\mathsf{rk}(\underline{L})}{2}-2} T_0\left(\frac{\ell}{sd^2}\right)\varphi, & \mathsf{rk}(\underline{L}) \text{ even} \end{cases}$$

- they map $J_{k,\underline{L}}$ to itself and they *preserve* cusp forms and Eisenstein series

- they are *Hermitian*: $\langle T(\ell)\varphi, \psi\rangle = \langle \varphi, T(\ell)\psi\rangle$
- they *commute*:

$$T(\ell)T(m) = \begin{cases} \sum_{d|(\ell,m)} d^{2k-\mathsf{rk}(\underline{L})-2} T\left(\ell m/d^2\right)\varphi, & \mathsf{rk}(\underline{L}) \text{ odd} \\ \sum_{d|(\ell^2,m^2)} \chi_{\underline{L}}(d) d^{k-\frac{\mathsf{rk}(\underline{L})}{2}-1} T\left(\ell m/d\right)\varphi, & \mathsf{rk}(\underline{L}) \text{ even,} \end{cases}$$

- a well-known result from Linear Algebra implies the following:

Theorem

The space $S_{k,\underline{L}}$ has a basis of simultaneous eigenforms for all $T(\ell)$.

- let φ be an *eigenform* of all $T(\ell)$: $T(\ell)\varphi = \lambda(\ell)\varphi$; if $\mathsf{rk}(\underline{L})$ is odd, then

$$L(s,\varphi) := \sum_{\substack{\ell\in\mathbb{N} \\ (\ell,\mathrm{lev}(\underline{L}))=1}} \lambda(\ell)\ell^{-s} = \prod_{(p,\mathrm{lev}(\underline{L}))=1} \left(1 - p^{-2}\lambda(p) + p^{2k-\mathsf{rk}(\underline{L})-2-2s}\right)^{-1}$$

- correspondence for $\mathsf{rk}(\underline{L})$ even:

$$J_{k,\underline{L}} \rightsquigarrow M_{k-\frac{\mathsf{rk}(\underline{L})}{2}}(?, \chi_{\underline{L}}\xi)$$

- let $r \in \mathrm{Iso}(\underline{L})$ and let χ be a primitive Dirichlet character modulo F for some $F \mid \mathrm{ord}(r)$; define the *twisted* Eisenstein series

$$E_{k,\underline{L},r,\chi} := \sum_{d \in (\mathbb{Z}/\mathrm{ord}(r)\mathbb{Z})^\times} \chi(d) E_{k,\underline{L},dr}$$

Theorem (Ajouz, 2015)

The twisted Eisenstein series $E_{k,\underline{L},r,\chi}$ (r as above modulo $(\mathbb{Z}/\mathrm{lev}(\underline{L})\mathbb{Z})^\times$ and χ as above with $\chi(-1) = (-1)^k$) form a system of Hecke eigenforms for $J^{\mathrm{Eis}}_{k,\underline{L}}$ with eigenvalues

$$\lambda(\ell) = \begin{cases} \sigma^{\chi,\overline{\chi}}_{2k-\mathrm{rk}(\underline{L})-2}(\ell), & \mathrm{rk}(\underline{L}) \text{ odd} \\ \overline{\chi(\ell)}\sigma^{\chi,\chi_{\underline{L}}}_{k-\frac{\mathrm{rk}(\underline{L})}{2}-1}(\ell^2), & \mathrm{rk}(\underline{L}) \text{ even.} \end{cases}$$

- the *discriminant module* associated to $\underline{L}$ is $D_{\underline{L}} := (L^{\#}/L, \beta \bmod \mathbb{Z})$
- the *orthogonal group* of $D_{\underline{L}}$ is $O(D_{\underline{L}}) := \{s : D_{\underline{L}} \xrightarrow{\sim} D_{\underline{L}} : \beta \circ s = \beta\}$
- Ajouz (2015): there exists an action of $O(D_{\underline{L}})$ on $J_{k,\underline{L}}$

$$(s, \varphi) \mapsto W(s)\varphi(\tau, \mathfrak{z}) := \sum_{D,t} C_{\varphi}\left(D, s(t)\right) e\left((\beta(t) - D)\tau + \beta(t, \mathfrak{z})\right)$$

- these operators *commute* with $T(\ell)$; therefore $W(s)\varphi$ and φ have the same Hecke eigenvalues if φ is an eigenform
- an element α in $L^{\#}/L$ such that $\mathrm{ord}(\alpha) = \mathrm{lev}(\alpha)$ or $2\mathrm{ord}(\alpha) = \mathrm{lev}(\alpha)$ is called *admissible*
- for every admissible α, consider the *reflection map*

$$s_{\alpha}(t) := t - \frac{\beta(t, \alpha)}{\beta(\alpha)}\alpha$$

Lemma

The maps s_{α} are elements of $O(D_{\underline{L}})$ and involutions.

Example

Skoruppa (1984) defines an operator on $J_{k,m}$ for every $n \parallel m$:

$$W_n\varphi(\tau, z) = \sum_{D,t} C(D, \lambda_n t) e\left((mt^2 - D)\tau + 2tz\right),$$

where λ_n is uniquely determined modulo $2m$ by the modular equations

$$\lambda_n \equiv -1 \bmod 2n$$
$$\lambda_n \equiv 1 \bmod \frac{2m}{n}.$$

- it can be shown that $W_n = W(s_\alpha)$ for some α in $\frac{1}{2m}\mathbb{Z}/\mathbb{Z}$
- the W_n are called Atkin–Lehner involutions by Skoruppa–Zagier (1988), because

$$\mathrm{tr}\left(T(l) \circ W_n, J_{k,m}\right) = \mathrm{tr}\left(T(l) \circ W_n, \mathfrak{M}^-_{2k-2}(m)\right)$$

Proposition (M. 2018)

The operators $W(s)$ are unitary for all s in $O(D_{\underline{L}})$ and they are Hermitian when s is a reflection.

- Eichler–Zagier (1985):

$$V(\ell) : \varphi(\tau, z) \mapsto \frac{1}{\ell} \sum_{ad=\ell} a^k \sum_{b \bmod d} \varphi\left(\frac{a\tau + b}{d}, az\right)$$

  maps $J_{k,m}$ to $J_{k,m\ell}$ ($\ell \in \mathbb{N}$)
- Gritsenko (1988) constructs these operators for arbitrary lattice index $\underline{L}$ using the embedding of $J_{k,\underline{L}}$ into $M_k(O^J)$:
  - remember elliptic Hecke operators

$$T(\ell) := \sum_{\substack{ad=\ell \\ a|d}} \mathrm{SL}_2(\mathbb{Z}) \left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right) \mathrm{SL}_2(\mathbb{Z})$$

  - set

$$G^J := \left\{ \begin{pmatrix} A^* & X_1 & T \\ 0 & E_{\mathsf{rk}(\underline{L})} & X \\ 0 & 0 & A \end{pmatrix} \in SO_{\mathbb{Q}}(\underline{M}) : \det(A) > 0 \right\}$$

  - consider the following embedding of $\mathcal{H}(\mathrm{SL}_2(\mathbb{Z}), \mathrm{M}_2^+(\mathbb{Z}))$ into $\mathcal{H}(O^J, G^J)$:

$$\iota : \mathrm{SL}_2(\mathbb{Z}) A \; \mathrm{SL}_2(\mathbb{Z}) \mapsto O^J \mathrm{diag}\left(\left(\det(A)^{-1} A\right)^*, E_{\mathsf{rk}(\underline{L})}, \det(A)^{-1} A\right) O^J$$

- if $O^J g O^J = \sum_i O^J g_i$, then set

$$\varphi|\left(O^J g O^J\right)(\tau,\mathfrak{z}) := \sum_i \Phi|_k g_i(Z) e\left(\frac{-\omega}{\det(A)}\right)$$

Proposition (Gritsenko, 1988)

If $\varphi \in J_{k,\underline{L}}$ and $O^J g O^J$ is an element of $\mathcal{H}(O^J, G^J)$, then

$$\varphi|\left(O^J g O^J\right) \in J_{k,\underline{L}(1/\det(A))}.$$

- we have $V(\ell) = \frac{1}{\ell}\iota(T(\ell))$

Corollary

It follows that $V(\ell)$ maps $J_{k,\underline{L}}$ to $J_{k,\underline{L}(\ell)}$, raising the level *by a factor of ℓ. Furthermore, the $V(\cdot)$ operators commute.*

- the $V(\cdot)$ operators are precisely the ones used for additive liftings:
  - $\varphi \in J_{k,\underline{L}} \implies \sum_{\ell \geq 1} V(\ell)\varphi(\tau,\mathfrak{z})e(\ell\omega) \in M_k\left(\tilde{O}^+(\underline{M})\right)$
  - $\varphi \in J_{k,1} \implies \sum_{\ell \geq 1} V(\ell)\varphi(\tau,z)e(\ell w) \in M_k^*(\Gamma_2)$
- note that

$$V(\ell)\varphi(\tau,\mathfrak{z}) = \sum_{\substack{n\in\mathbb{Z},t\in L^\# \\ \ell n \geq \beta(t)}} \sum_{\substack{d|(n,\ell) \\ \frac{t}{d}\in L^\#}} d^{k-1} c_\varphi\left(\frac{nl}{d^2},\frac{t}{d}\right) e\left(n\tau + \beta(t,\mathfrak{z})\right),$$

- in particular, they *preserve* cusp forms and Eisenstein series

- Eichler–Zagier (1985): $U(\ell) : \varphi(\tau, z) \mapsto \varphi(\tau, \ell z)$ maps $J_{k,m}$ to $J_{k,m\ell^2}$
- let $\underline{L} = (L, \beta)$ and $\underline{L}' = (L', \beta')$ be two lattices; an *isometry* of $\underline{L}$ into $\underline{L}'$ is an *injective* linear map $\sigma : L \otimes_{\mathbb{Z}} \mathbb{Q} \to L' \otimes_{\mathbb{Z}} \mathbb{Q}$ such that

$$\beta' \circ \sigma = \beta \qquad \text{and} \qquad \sigma \underline{L} \subseteq \underline{L}'$$

Definition

For every isometry σ of $\underline{L}$ into $\underline{L}'$, define a linear operator $U(\sigma)$ from $J_{k,\underline{L}'}$ to $\mathrm{Hol}(\mathfrak{H} \times (L \otimes_{\mathbb{Z}} \mathbb{C}) \to \mathbb{C})$ as

$$U(\sigma)\varphi(\tau, z) := \varphi(\tau, \sigma(z)).$$

Proposition (M. 2018)

When $L \otimes_{\mathbb{Z}} \mathbb{Q} \simeq L' \otimes_{\mathbb{Z}} \mathbb{Q}$, the operator $U(\sigma)$ maps $J_{k,\underline{L}'}$ to $J_{k,\underline{L}}$.

- for example, the map $\sigma_\ell : \mathbb{Q} \to \mathbb{Q}, \sigma_\ell(x) = \ell x$ is an isometry of $(\mathbb{Z}, (x, y) \mapsto 2m\ell^2 xy)$ into $(\mathbb{Z}, (x, y) \mapsto 2mxy)$ and $U(\sigma_\ell) = U(\ell)$ maps $J_{k,m}$ to $J_{k,m\ell^2}$

- the definition of the dual lattice implies that $\mathrm{lev}(\underline{L}') \mid \mathrm{lev}(\underline{L})$ and it follows that $U(\sigma)$ *raises the level*
- when $\mathrm{rk}(\underline{L}) = 1$, the level is always raised by a square factor, but for higher rank this need not be the case

- $(\sigma(L), \beta')$ is a sublattice of $\underline{L}'$ and $\underline{L} \simeq (\sigma(L), \beta')$; conversely, any sublattice (M, β') of $\underline{L}'$ gives rise to an isometry of (M, β') into $\underline{L}'$
- the previous Proposition can be rephrased as

Every Jacobi form of weight k and index $\underline{L}'$ is
a Jacobi form of weight k and index $\underline{M}$

- in particular, the U operators *preserve* cusp forms and Eisenstein series

Proposition (Nikulin, 1980)

Let $\underline{L} = (L, \beta)$ be a positive-definite, even lattice. Then there is a one-to-one correspondence between overlattices of $\underline{L}$ and isotropic subgroups of $D_{\underline{L}}$. For every such overlattice $\underline{L}' = (L', \beta)$, the correspondence is given by

$$\underline{L}' \mapsto L'/L.$$

- for every isotropic subgroup I of $D_{\underline{L}}$, set $\underline{L}_I := (L + I, \beta)$
- we obtain an operator $U(I) = U(\iota_{\underline{L}, \underline{L}_I})$ which acts as an inclusion map of $J_{k,\underline{L}_I}$ into $J_{k,\underline{L}}$

Theorem (M. 2018)

The operators $U(\cdot)$ and $V(\cdot)$ commute with each other. They commute with the Hecke operators. Furthermore, if an admissible element α in $L^{\#}/L$ is such that $(\mathrm{lev}(\alpha), \ell) = 1$, then $V(\ell)W(s_\alpha) = W(s_\alpha)V(\ell)$. If I is an isotropic subgroup of $D_{\underline{L}}$ and α is admissible both in $L^{\#}/L$ and in $L_I^{\#}/L_I$, then $U(I)W(s_\alpha) = W(s_\alpha)U(I)$.

Definition

An *oldform* in $J_{k,\underline{L}}$ is an element φ which is either equal to $V(\ell)\psi$ for some ℓ in $\mathbb{N}$ and some ψ in $J_{k,\underline{L}(1/\ell)}$ or it is an element of $J_{k,\underline{L}_I}$ for some isotropic subgroup I of $D_{\underline{L}}$.

Lemma

If $\varphi \in J_{k,\underline{L}}$ is such that $C_\varphi(D, r) = 0$ for r not it $L_I^{\#}$, then φ is an oldform.

Proposition (M. 2018)

The following holds for every s in $O(D_{\underline{L}})$:

$$W(s)E_{k,\underline{L},r,\chi} = E_{k,\underline{L},s^{-1}(r),\chi}.$$

Proposition (M. 2018)

The following holds for every ℓ in $\mathbb{N}$ and every isotropic I in $D_{\underline{L}}$:

$$V(\ell)U(I)E_{k,\underline{L}_I,r} = \sum_{\substack{t \in L(\ell)^{\#}/L \\ \ell\beta(t) \in \mathbb{Z}}} \sum_{d \mid (\ell\beta(t),\ell)} d^{k-1} \delta_{\underline{L}_I}\left(r, \frac{\ell t}{d}\right) E_{k,\underline{L}(\ell),t}$$

Proposition

If $F \neq \operatorname{ord}(r)$, then $E_{k,\underline{L},r,\chi}$ is an oldfrom.

To do

- is the above criterion for oldforms *exhaustive*?
- are there more $V(\cdot)$ operators?
  - guess: no
- do we have *Multiplicity One*?
- there should be more *old* Eisenstein series
  - when rk($\underline{L}$) $= 1$, all Eisenstein series are old if m is not a square and, if $m = f^2$, then only $E_{k,f^2,\frac{1}{f},\chi}$ is new

Thank you!