Jacobi forms and Number Theory

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Introduction

Number Theory is the study of mathematical objects related to the *integer numbers* $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, with applications in other areas of Mathematics, Physics and Computer Science.

Automorphic forms are a central object studied in Number Theory. They are *functions* taking values in a vector space such as the *complex numbers* $\mathbb{C} = \{a + ib : a, b \in \mathbb{R}, i^2 = -1\}$, where $\mathbb{R}$ is the set of real numbers. Their main feature is that they satisfy a property similar to *periodicity*.

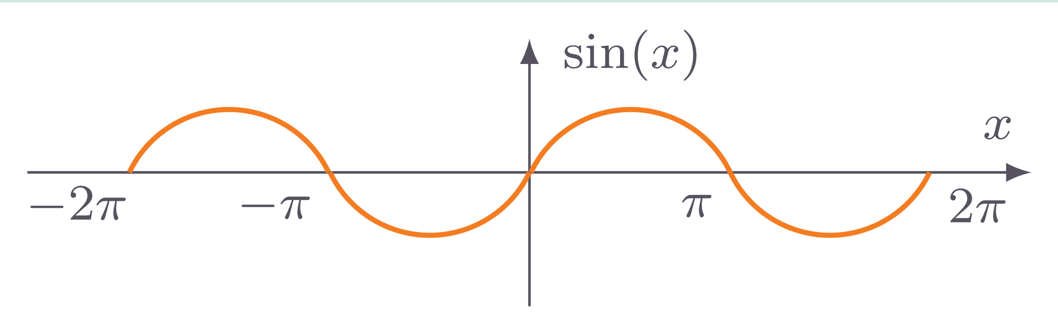

Figure 1: periodicity of sin

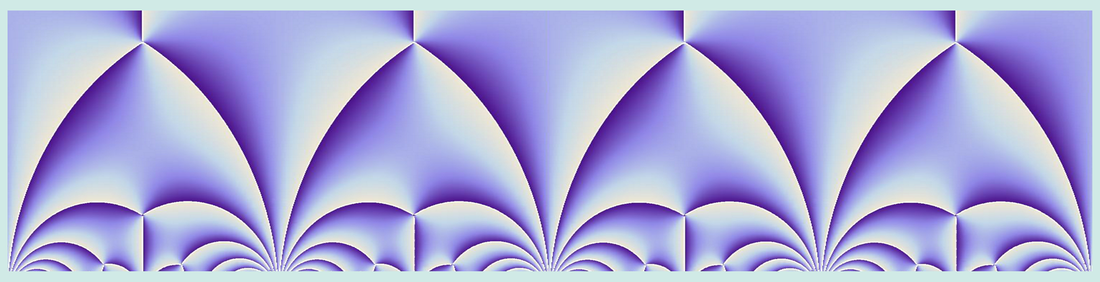

Figure 2: transformation of j-function

Jacobi forms

Amongst different types of automorphic forms, **Jacobi forms** are an elegant *intermediate* between elliptic modular forms (studied extensively) and Siegel modular forms (a higher dimensional generalization).

They are parametrized by *weight* (an integer) and *index* (a lattice). They are functions of *two variables*: $\tau = a+bi$ with $b > 0$ and $\mathfrak{z} = (z_1, \dots, z_n)$, a vector with each z_i in $\mathbb{C}$ and where n is the *rank* of the lattice. For fixed weight k and index $\underline{L}$, they form a *vector space* $J_{k,\underline{L}}$, an infinite family so to speak.

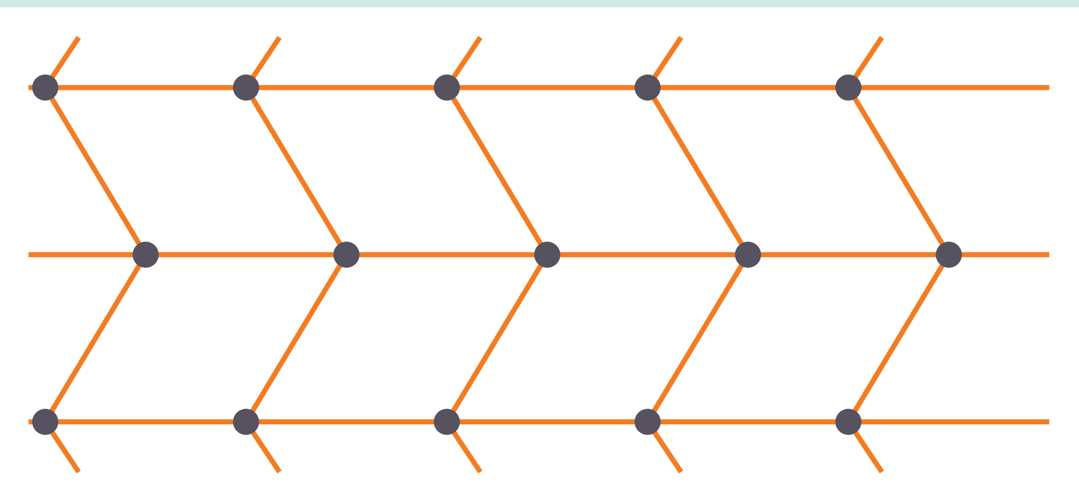

Figure 3: example of lattices in $2D$

Linear operators

Jacobi forms are particularly 'nice' because each $J_{k,\underline{L}}$ is a finite dimensional **Hilbert space**. Such spaces have a *basis*: there is a finite number of elements in $J_{k,\underline{L}}$, $\{\varphi_1, \varphi_2, \dots, \varphi_t\}$, such that every other $\varphi \in J_{k,\underline{L}}$ can be written as:

$$\varphi(\tau, \mathfrak{z}) = \sum_{i=1}^{t} \lambda_i \varphi_i(\tau, \mathfrak{z}),$$

with $\lambda_i \in \mathbb{C}$. So we no longer need to think about *infinitely many* elements and life becomes much easier!

Linear operators are crucial in making a vector space nice. We defined the *level raising* operators $V(l)$ (l is any positive integer) to map $\varphi(\tau, \mathfrak{z})$ to:

$$l^{\frac{k}{2}-1} \sum_{\substack{M \in \Gamma \backslash \mathcal{M}_2(\mathbb{Z}) \\ \det(M) = l}} U(\sqrt{l}) \left(\varphi|_{k,\underline{L}} M\right)(\tau, \mathfrak{z}),$$

where $U(l)\varphi(\tau, \mathfrak{z}) = \varphi(\tau, l\mathfrak{z})$. Level raising means that $V(l)$ maps a Jacobi form $\varphi \in J_{k,\underline{L}}$ to one in $J_{k,\underline{L}'}$, where $\underline{L}'$ has *level* equal to l times the level of $\underline{L}$.

Any periodic function is uniquely defined by its *Fourier expansion*. For example, if the Fourier expansion of φ is

$$\varphi(\tau, \mathfrak{z}) = \sum_{r \in L^\#} \sum_{\substack{n \in \mathbb{Z} \\ n \geq \beta(r)}} c(n, r) e(n\tau + \beta(r, \mathfrak{z})),$$

then

$$V(l)\varphi(\tau, \mathfrak{z}) = \sum_{\substack{n \in \mathbb{Z}, r' \in L'^\# \\ n \geq \beta'(r')}} \sum_{\substack{a | (n,l) \\ \frac{r'}{a} \in L'^\#}} a^{k-1} c\left(\frac{nl}{a^2}, \frac{lr'}{a}\right) e(n\tau + \beta'(r', \mathfrak{z})).$$

We also described how these operators interact with each other:

$$V(l) \circ V(l') = \sum_{d | \gcd(l, l')} d^{k-1} U(d) \circ V\left(\frac{ll'}{d^2}\right).$$

Poincaré and Eisenstein series

Another way that to break down our space of Jacobi forms is into *cusp forms* and non-cusp forms. These spaces are *perpendicular* and their intersection is empty:

$$J_{k,\underline{L}} = J_{k,\underline{L}}^{\text{cusp}} \oplus J_{k,\underline{L}}^{\text{Eis}}.$$

Cusp forms are generated by **Poincaré series** and non-cusp forms are generated by **Eisenstein series**. Using multi-dimensional complex integrals and contour integration, we computed the Fourier expansions of both. For example:

$$P_{k,\underline{L},r,D}(\tau, \mathfrak{z}) = \sum_{\substack{n' \in \mathbb{Z}, r' \in L^\# \\ n' > \beta(r')}} G_{k,\underline{L},D,r}(n', r') e\left(n'\tau + \beta(r', \mathfrak{z})\right),$$

is the Fourier expansion of a Poincaré series, where

$$\begin{aligned} G_{k,\underline{L},D,r}(n', r') :=& \delta_{\underline{L}}(D, r, D', r') + (-1)^k \delta_{\underline{L}}(D, r, D', -r') + 2\pi i^k \\ & \times \det(\underline{L})^{-\frac{1}{2}} (D'/D)^{\frac{k}{2} - \frac{\mathrm{rk}(\underline{L})}{4} - \frac{1}{2}} \cdot \sum_{c \geq 1} \left(H_{\underline{L},c}(n, r, n', r')\right. \\ & \left. + (-1)^k H_{\underline{L},c}(n, r, n', -r')\right) \cdot J_{k - \frac{\mathrm{rk}(\underline{L})}{2} - 1}\left(4\pi (DD')^{\frac{1}{2}}/c\right), \end{aligned}$$

and all of the quantities appearing in this formula are explicitly known. We also showed that, as expected, computing the *Petersson scalar product* of a cusp form φ and a Poincaré series reproduces its Fourier coefficients:

$$\langle \varphi, P_{k,\underline{L},r,D} \rangle = \lambda_{k,\underline{L},D} \cdot c(n, r),$$

where $\lambda_{k,\underline{L},D}$ is an explicit constant.

Future work

The operators $U(l)$ and $V(l)$ are a first step in the direction of a *theory of newforms* for Jacobi forms. **Newforms** are those forms in a fixed space $J_{k,\underline{L}}$ which don't 'come from' a space with lower level (if they do, we call them **oldforms**). This gives us yet another way of decreasing the number of things we have to work with.

The results on Poincaré and Eisenstein series give us a **better understanding** of the spaces of Jacobi forms and enable us to do more mathematical work with them. The 'shape' of the Fourier expansions we have obtained offers great support for an explicit correspondence between Jacobi forms and elliptic modular forms, for example.