INTRODUCTION TO MODULAR FORMS UON MATHEMATICS STUDENT THEORY

ANDREEA MOCANU

Number Theory as a subject started off in the Antiquity as the study of natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$. Then it became the study of integer numbers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ and now it has become the study of any mathematical object that is *related* to integer numbers.

This is why Number Theory has applications in many other areas of Maths, such as Group Theory, Galois Theory, Representation Theory, Algebraic Geometry, but also outside of Maths, such as Cryptography, Computer Science and Mathematical Physics.

The purpose of this talk is to discuss modular forms, which are functions f defined on the upper half-plane, $\mathfrak{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$. They are functions which are intimately related to integer numbers, as we shall see in their definition:

Definition 0.1 (Modular form)**.** A modular form of integer *weight* k with respect to $\mathrm{SL}_2(\mathbb{Z})$ is a functions as above which satisfies the following properties:

- f is holomorphic.
- For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ (the group of 2×2 matrices with determinant $ad - bc = 1$), we have

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau),$$

  for any $\tau \in \mathfrak{H}$.

These functions form a $\mathbb{C}$-vector space for fixed k, which is usually denoted by $M_k(\ \mathrm{SL}_2(\mathbb{Z}))$. In particular, this vector space is *finite dimensional*, so it has a basis and any modular form of weight k is a combination of elements in that basis. We could have defined modular forms with respect to any Fuchsian group $G \subseteq \mathrm{SL}_2(\mathbb{Z})$ (i.e. a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$).

We can map the upper half-plane to the Poincaré disk (i.e. the disk of radius 1 centered at the origin) via the map $\tau \mapsto e^{2\pi i \tau}$. Thus, every modular form f determines uniquely a function $g(q) = f(\tau)$, where

$q = e^{2\pi i\tau}$. The function g then has a *Fourier expansion*

$$g(q) = \sum_{n\geq 0} a_n q^n,$$

and we call this the Fourier expansion of f as well. Thus, every modular forms determines a sequence $(a_n)_{n\in\mathbb{N}}$, the sequence of its Fourier coefficients and it is in fact uniquely determined by this sequence. Thus, we get another connection with the integer numbers.

Conversely, if we are given a sequence $(a_n)_{n\in\mathbb{N}}$, we could ask ourselves: *Is its generating function of modular form?* If the answer is yes, then we can use the vector space structure of $M_k(G)$ to find interesting and useful information about our sequence. For a sequence $(a_n)_{n\in\mathbb{N}}$, its generating functions is simply

$$f(\tau) = \sum_{n\geq 0} a_n e^{2\pi i\tau}.$$

Let us consider the following:

Example 0.1 (Sums of squares)**.** Given positive integers m and n, we want to count the number of integral solutions of the equation

$$\sum_{i=1}^{m} x_i^2 = n.$$

In other words, we are looking for the number $r_m(n)$ of ways of writing n as a sum of m squares. This was investigated by a number of famous mathematicians, such as Fermat, Jacobi and Lagrange. For example, when $m = 1$, we will have $r_1(0) = 1$, $r_1(n) = 0$ if n is not a square and $r_1(n) = 2$ for a square number different from zero.

It turns out that if we define

$$\theta(\tau) = \sum_{j\in\mathbb{Z}} q^{j^2} = \sum_{n\geq 0} r_1(n) q^n,$$

then θ^2 is a modular form of weight 1, θ^{2k} is a modular form of weight k and, in general, θ^m is the generating function of $r_m(n)$.

Using known facts about spaces of modular forms, we can write down formulas of the type

$$r_2(n) = 4\sum_{d|n} \chi(n)$$

$$r_4(n) = 8(2 + (-1)^n)\sum_{\substack{d|n \\ 2\nmid d}} d$$

$$r_6(n) = 16\sum_{d|n} \chi\left(\frac{n}{d}\right) d^2 - 4\sum_{d|n} \chi(d)d^2,$$

and so on, where $\chi(n)$ is a Dirichlet character which is equal to

$$\begin{cases} 1, & n = 4k+1 \\ -1, & n = 4k+3 \\ 0, & n = 2k. \end{cases}$$